

Random experiments

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1.1 **Events**

1.2 **Probability**

1.3 **Conditional probability**

1.4 **Bayes' formula**

1.5 **Independence**

1.6 **Combinatorics**

Deterministic vs. *random* experiments.

An experiment is *random* if although it is repeated in the same manner every time, can result in different outcomes:

- The set of all possible outcomes is completely determined before carrying it out.
- Before we carry it out, we cannot predict its outcome.
- It can be repeated as many times as we want always under the same conditions (leading to different outcomes).

1.1 Events (sets)

The **sample space** is the set of all possible outcomes of a random experiment, we will denote it by S .

An **event** is a subset of the sample space (any set of outcomes of the random experiment).

Special events

The **certain event**, S , always occurs. The **null (impossible) event**, \emptyset , does never occur.

1.1 Events (sets)

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Special events

The **certain event**, S , always occurs. The **null (impossible) event**, \emptyset , does never occur.

Example

Random experiment: toss three coins.

Sample space: $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Some events: $A =$ “exactly two heads” $= \{HHT, HTH, THH\}$;

$B =$ “at least two heads” $= \{HHT, HTH, THH, HHH\}$.

Package prob

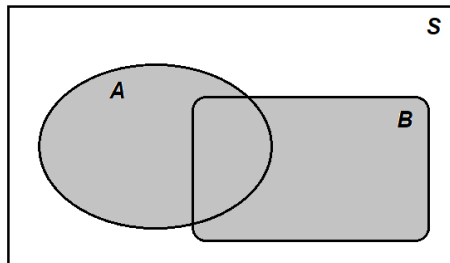
```
install.packages('prob', repos='http://cran.rediris.es/')
```

```
library(prob)  
tosscoin(3)
```

```
##   toss1 toss2 toss3  
## 1     H     H     H  
## 2     T     H     H  
## 3     H     T     H  
## 4     T     T     H  
## 5     H     H     T  
## 6     T     H     T  
## 7     H     T     T  
## 8     T     T     T
```

Event (set) operations

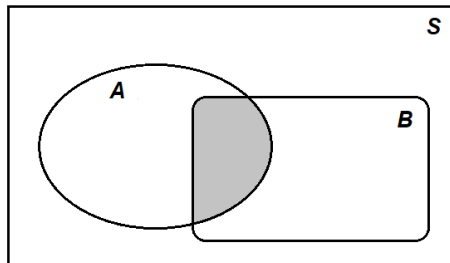
- The **union** of two events A and B , denoted by $A \cup B$ occurs when either of the two events (or both of them simultaneously) do occur. Represented by the grammatical expression A or B .



$$A \cup B$$

Event (set) operations

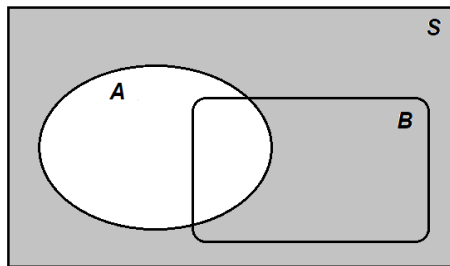
- The **intersection** of two events A and B , denoted by $A \cap B$ occurs when both of them do simultaneously occur. It is represented by the grammatical expression A and B .



$$A \cap B$$

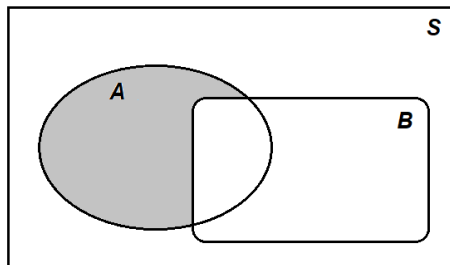
Event (set) operations

- The **complementary event** to A , denoted by \bar{A} (read *not A*) occurs when A does not occur.



Event (set) operations

- The **set different** of A and B denoted by $A \setminus B$ occurs when A does occur, but B does not.



$$A \setminus B = A \cap \bar{B}$$

Event (set) operations

```
S=probspace(tosscoin(2))
A=subset(S,toss1=="H")
B=subset(S,toss2=="H")
intersect(A,B)
```

```
##  toss1 toss2 probs
##  1      H      H  0.25
```

```
union(A,B)
```

```
##  toss1 toss2 probs
##  1      H      H  0.25
##  2      T      H  0.25
##  3      H      T  0.25
```

Algebra of events (some properties of the ops. with events)

General properties

- **Commutative:** $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- **Associative:** $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Neutral element:** $A \cup \emptyset = A = A \cap S$.
- **Complementation:** $A \cup \bar{A} = S$ and $A \cap \bar{A} = \emptyset$.
- **Idempotence:** $A \cup A = A$ and $A \cap A = A$.
- **Absorption:** $A \cup S = S$ and $A \cap \emptyset = \emptyset$.
- **Simplification:** $A \cap (A \cup B) = A = A \cup (A \cap B)$.

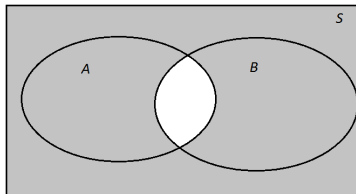
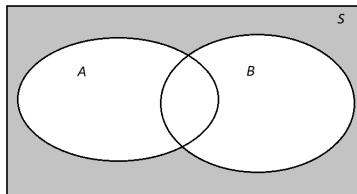
Distributive laws

- **Union wrt intersection:** $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
- **Intersection wrt union:** $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Algebra of events (some properties of the ops. with events)

De Morgan's laws

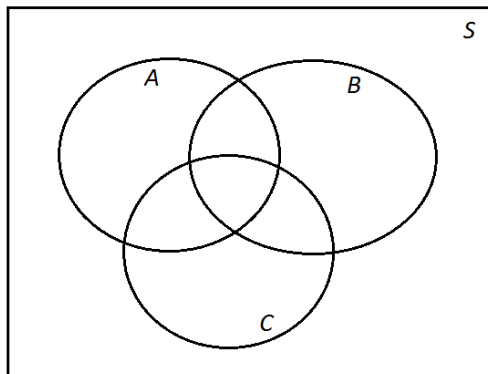
- $\overline{A \cup B} = \bar{A} \cap \bar{B}$;
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$.



Grammatical expressions and events

Given three events A, B, C write as union and intersections

- A and B occur, but C does not.
- Exactly two of the three events occur.
- At most two of the three events occur.



1.2 Probability

A probability P assesses a number to every event A associated with the random experiment. It is interpreted as the likelihood (or chance) that A occurs.

Probability axioms

- **Nonnegativity:** $P(A) \geq 0$ for every event A .
- **Additivity:** If A_i with $i \in I$ denumerable are events and $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$P\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} P(A_i).$$

- **Normalization:** $P(S) = 1$.

(The structure of the family of events at with a probability is defined is known as σ -algebra. A σ -algebra 1. contains the certain event S , 2. is closed under complementation, and 3. is closed under countable unions.)

Probability

```
S=probspace(tosscoin(2))  
A=subset(S,toss1=="H")  
B=subset(S,toss2=="H")  
Prob(A)
```

```
## [1] 0.5
```

```
Prob(intersect(A,B))
```

```
## [1] 0.25
```

```
probspace(tosscoin(1),probs=c(1,2))
```

```
##      toss1      probs  
## 1      H 0.3333333  
## 2      T 0.6666667
```

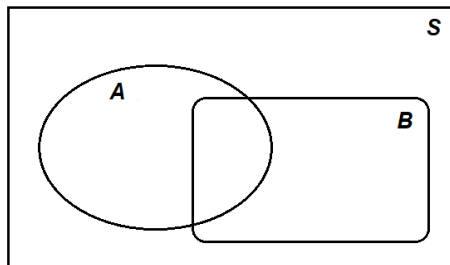


```
S=probspace(tosscoin(1),probs=c(1,2))  
set.seed(1)  
empirical(sim(S,ntrials=1000))
```

```
##   toss1 probs  
## 1      H 0.334  
## 2      T 0.666
```

Properties of a probability

- $P(\bar{A}) = 1 - P(A)$.
- $P(\emptyset) = 0$.
- If $A \subset B$, then $P(A) \leq P(B)$.
- $P(A \setminus B) = P(A) - P(A \cap B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) \dots$



Uniform discrete probability

Laplace's rule

If a random experiment can result in a finite number of equally likely outcomes, the probability of some event A is

$$P(A) = \frac{\text{number of outcomes favourable to } A}{\text{number of possible outcomes}}.$$

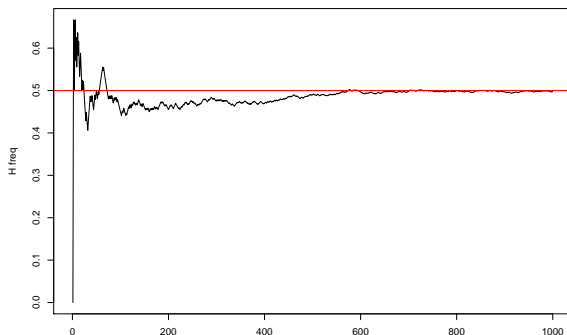
Example

Random experiment: toss a 6-face *fair* die

$$P(\text{'even outcome'}) = \frac{3}{6} = \frac{1}{2}.$$

How to interpret a probability (LLN)

```
set.seed(2)
S=probspace(tosscoin(1))
plot(cumsum(sim(S,ntrials=1000)=="H")/(1:1000),
     type="l",ylab="H freq",xlab="n toss")
abline(h=0.5,col="red")
```

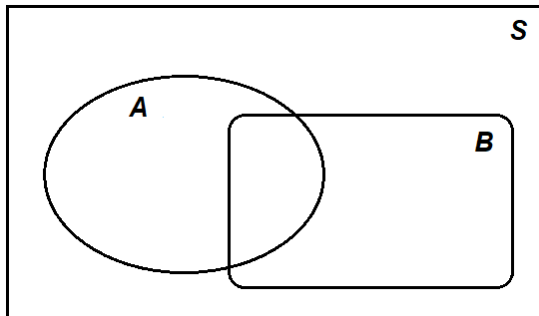


1.3 Conditional probability

Conditional probability

The **conditional probability** of A given B is the probability that A occurs given that B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$



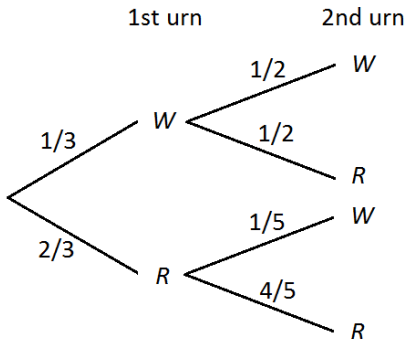
Conditional probability

```
S=probspace(tosscoin(3))
H1=subset(S,toss1=="H")
H2=subset(S,toss2=="H")
H3=subset(S,toss3=="H")
Prob(H1,given=intersect(H2,H3))
```

```
## [1] 0.5
```

Conditional probability (example)

An urn contains 1 white ball and 2 red balls. One ball is taken from the urn at random. If it is a white ball, it is returned into the urn together with another white ball. If it is a red ball, it is returned together with other 2 red balls. In a second stage, another ball is taken from the urn at random.



Conditional probability (example)

An urn contains 1 white ball and 2 red balls. One ball is taken from the urn at random. If it is a white ball, it is returned into the urn together with another white ball. If it is a red ball, it is returned together with other 2 red balls. In a second stage, another ball is taken from the urn at random.

Events

- $W_1 \equiv$ 'first ball is white'
- $R_1 \equiv$ 'first ball is red'
- $W_2 \equiv$ 'second ball is white'
- $R_2 \equiv$ 'second ball is red'

Probabilities

- $P(W_1) = 1/3$; $P(R_1) = 2/3$
- $P(W_2|W_1) = 1/2$; $P(R_2|W_1) = 1/2$
- $P(W_2|R_1) = 1/5$; $P(R_2|R_1) = 4/5$

Properties of the conditional probability

- **Nonnegativity:** $P(A|B) \geq 0$ for every event A .
- **Additivity:** If A_i with $i \in I$ denumerable are events and $A_i \cap A_j \cap B = \emptyset$ whenever $i \neq j$, then

$$P\left(\bigcup_{i \in I} A_i \mid B\right) = \sum_{i \in I} P(A_i | B).$$

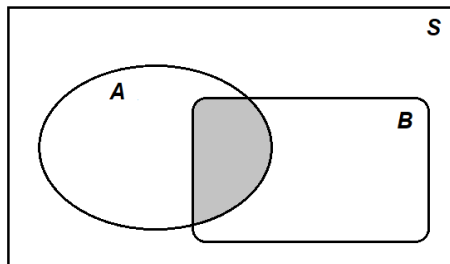
- **Normalization:** $P(S|B) = 1$, actually $P(B|B) = 1$.
- and all other properties of a probability, e.g. $P(\bar{A}|B) = 1 - P(A|B)$.

1.4 Bayes' formula

Multiplication rule

Given n events A_1, A_2, \dots, A_n , the probability that all of them occur simultaneously is

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1) \dots P(A_n|A_1 \cap \dots \cap A_{n-1}).$$



Multiplication rule

An urn contains 1 white ball and 2 red balls. One ball is taken from the urn at random. If it is a white ball, it is returned into the urn together with another white ball. If it is a red ball, it is returned together with other 2 red balls. In a second stage, another ball is taken from the urn at random.

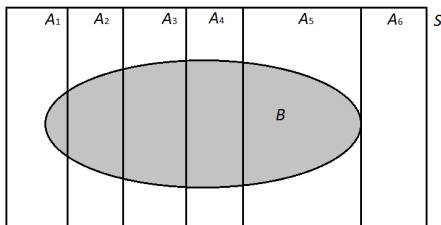
Two balls are extracted from the urn. What is the probability that both of them are red?

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}.$$

Total probability rule

Let A_1, \dots, A_n be mutually exclusive (disjoint) events whose union is the whole of the sample space (partition) and assume $P(A_i) > 0$ for every i . For every event B we have

$$\begin{aligned} P(B) &= P(A_1 \cap B) + \dots + P(A_n \cap B) \\ &= P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n). \end{aligned}$$



Total probability rule (example)

An urn contains 1 white ball and 2 red balls. One ball is taken from the urn at random. If it is a white ball, it is returned into the urn together with another white ball. If it is a red ball, it is returned together with other 2 red balls. In a second stage, another ball is taken from the urn at random.

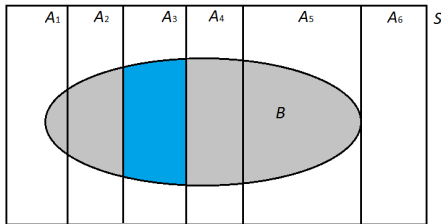
What is the probability that the second ball extracted from the urn is red?

$$P(R_2) = P(W_1)P(R_2|W_1) + P(R_1)P(R_2|R_1) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{4}{5} = 0.7.$$

Bayes' rule

Let A_1, \dots, A_n be mutually exclusive (disjoint) events whose union is the whole of the sample space (partition) and assume $P(A_i) > 0$ for every i . For every event B with $P(B) > 0$ we have

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\ &= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)}. \end{aligned}$$



Bayes' rule (example)

An urn contains 1 white ball and 2 red balls. One ball is taken from the urn at random. If it is a white ball, it is returned into the urn together with another white ball. If it is a red ball, it is returned together with other 2 red balls. In a second stage, another ball is taken from the urn at random.

If the second ball extracted from the urn is red, what is the probability that the first one was also red?

$$P(R_1|R_2) = \frac{P(R_1)P(R_2|R_1)}{P(R_2)} = \frac{\frac{2}{3} \cdot \frac{4}{5}}{0.7} = 0.762.$$

1.5 Independence

Independent events

Two events A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

In plain words, the fact that A occurs does not contain any information about the possible occurrence of B .

- For any event B with $P(B) > 0$, the events A and B are independent iff $P(A|B) = P(A)$.
- If A and B are independent, so are A and \bar{B} (also are the pair \bar{A} and B and the pair \bar{A} and \bar{B}).

More about independence

Conditional independence

A and B are **conditionally independent** given C with $P(C) > 0$ if

$$P(A \cap B | C) = P(A | C)P(B | C).$$

- equivalently $P(A | B \cap C) = P(A | C)$.

Examples

- **Random experiment:** Toss 2 fair coins (indep. but not cond. indep.)
Events: $H_1 \equiv$ 'Head coin 1'; $H_2 \equiv$ 'Head coin 2'; $E \equiv$ 'Equal outcomes'.
- **Random experiment:** Select at random a coin from a fair one and another with two heads. Afterwards toss it twice (cond. indep. but not indep.)
Events: $H_1 \equiv$ 'Head coin 1'; $H_2 \equiv$ 'Head coin 2'; $F \equiv$ 'Fair coin'.

More about independence

Independence of a collection of events

The events A_1, A_2, \dots, A_n are **independent** if for any subset of them, $I \subset \{1, 2, \dots, n\}$

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i).$$

Example

- **Random experiment:** Toss 2 fair coins
Events: $H_1 \equiv$ 'Head coin 1'; $H_2 \equiv$ 'Head coin 2'; $E \equiv$ 'Equal outcomes'.

1.6 Combinatorics

The basic principle of counting (multiplication rule)

Two experiments are to be performed, if experiment 1 can result in any one of n_1 possible outcomes and if, for each outcome of experiment 1, there are n_2 possible outcomes of experiment 2, then together there are $n_1 \cdot n_2$ possible outcomes of the two experiments.

Example

An exam consists of two questions and there is a unique set of eight possible answers. In how many ways can the exam be completed? (each question is to be linked to a unique answer and no pair of questions can be linked to the same answer)

The generalized principle of counting (multiplication rule)

k experiments are to be performed, if experiment 1 can result in any one of n_1 possible outcomes and if, for each outcome of experiment 1, there are n_2 possible outcomes of experiment 2, ..., and for each outcome of the previous experiments there are n_k outcomes of experiment k , then together there are $n_1 \cdot n_2 \cdots n_k$ possible outcomes of the sequence of experiments.

Example

The first five digits of all phone numbers at uc3m are 91 624.

- What is the probability that a randomly chosen phone number at uc3m has no repeated digits?
- What is the probability that none of the four remaining digits is repeated in a randomly chosen uc3m phone number?

Permutations and k -permutations (sequences, order does matter)

Given n distinct objects:

- the number of possible sequences that can be formed with them is called the **permutations** of n elements, $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.
- the number of possible sequences of $1 \leq k \leq n$ that can be formed with them is called k -**permutations** of n elements, $n(n-1)\cdots(n-k+1) = n!/(n-k)!$ (by agreement $0! = 1$)

`factorial(n)`

Example

An exam consists of eight questions and there is a unique set of eight possible answers.

- In how many ways can the exam be completed?
- In how many ways can the first two questions be answered?

Combinations (sets, order does not matter)

Given n objects:

- the number of possible sets of $1 \leq k \leq n$ elements that be formed with them is the number of **combinations** of k out of n objects

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

`choose(n, k)`

Example

We toss a coin 5 times,

- what is the number of possible outcomes? (size of the sample space)
- how many outcomes contain exactly 2 heads?

Partitions

Given n objects:

- if there are r groups of objects, containing n_i objects each, the number of ways to arrange the n objects in a sequence is the number of **partitions** of n objects into r groups with the i -th group having n_i objects,

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Example

We toss a 4-face die 5 times,

- what is the number of possible outcomes? (size of the sample space)
- how many outcomes contain 2 ones, 2 twos, 1 three (and 0 fours)?