

# Continuous random variables

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## Outline

- 3.1 Density mass function and cdf
- 3.2 Mean, variance, and quantiles
- 3.3 Uniform distribution
- 3.4 Transformations of a random variable
- 3.5 Exponential distribution
- 3.6 Normal distribution

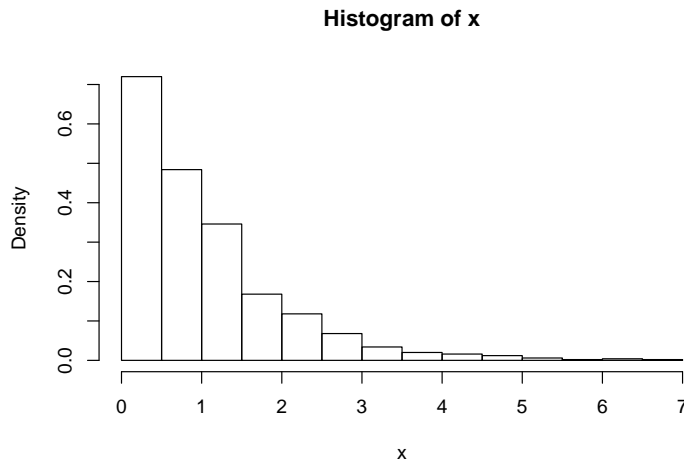
## Introduction

A **random variable**  $X : S \mapsto \mathbb{R}$  is **continuous** if its support  $X(S)$  contains an interval of real numbers, or more precisely if its probability law can be described in terms of a nonnegative real function  $f_X$  (**density mass function**) in such a way the the probability that  $X$  lies on any (borelian)  $A \subset \mathbb{R}$  can be computed as

$$P(X \in A) = \int_A f_X(x)dx .$$

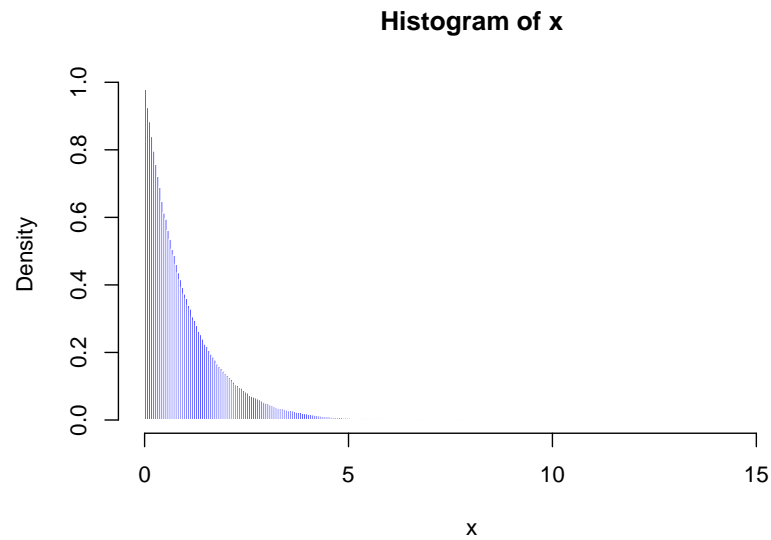
## Introduction (histogram, 1000 observations)

```
set.seed(1)
x=rexp(1000)
hist(x,probability=T)
```



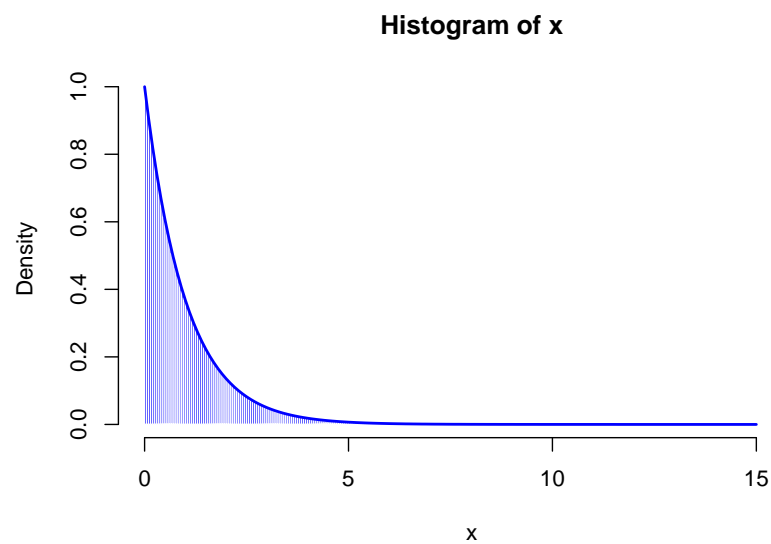
## Introduction (histogram, 1000000 observations)

```
set.seed(1)
x=rexp(1000000)
hist(x,nclass=250,probability=T,border="white",col="blue")
```



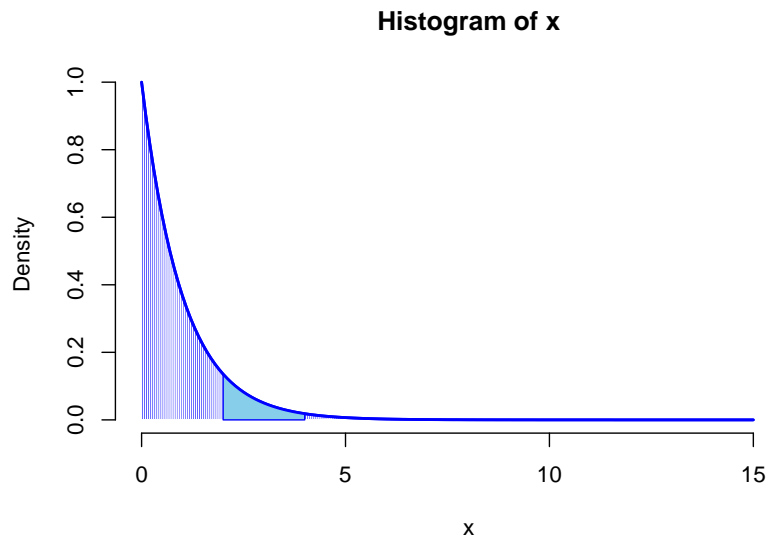
## Introduction (histogram and density)

```
hist(x,nclass=250,probability=T,border="white",col="blue")
t=seq(0,15,by=.1)
points(t,dexp(t),type="l",col="blue",lwd=2)
```



## Introduction (probability of an interval, $P(2 < X < 4)$ )

```
cord.x <- c(2,seq(2,4,0.01),4)
cord.y <- c(0,dexp(seq(2,4,0.01)),0)
polygon(cord.x,cord.y,col='skyblue')
points(t,dexp(t),type="l",col="blue",lwd=2)
```



### 3.1 Density mass function and cdf

#### Definition of density mass function

Every **density mass function**  $f : \mathbb{R} \mapsto \mathbb{R}$  satisfies 1.  $f(x) \geq 0$ ; 2.  $\int_{-\infty}^{+\infty} f(x)dx = 1$ .

#### Density mass function of a continuous r.v.

If  $X$  is a *continuous* r.v. and  $f_X$  its associated density mass function, the probability that  $X$  lies in any (borelian)  $A \subset \mathbb{R}$  is

$$P(X \in A) = \int_A f_X(x)dx.$$

When  $A = [a, b]$ , we have  $P(a \leq X \leq b) = \int_a^b f_X(x)dx$ .

### 3.1 Density mass function and cdf

#### Properties of continuous r.v.s

- $P(X = a) = \int_a^a f_X(x)dx = 0$  for any  $a \in \mathbb{R}$ ;
- $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$ .

## Example

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$P(2 < X < 4) = \int_2^4 e^{-x} dx = -(e^{-4} - e^{-2}) = 0.117.$$

## 3.1 Density mass function and cdf

### Cumulative distribution function, cdf

The **cumulative distribution function (cdf)** of r.v.  $X$  evaluated at  $x \in \mathbb{R}$  is the probability that  $X$  is not greater than  $x$ ,

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$$

### Properties of the cdf of a continuous random variable

- $\lim_{x \rightarrow -\infty} F(x) = 0$ ;
- $\lim_{x \rightarrow +\infty} F(x) = 1$ ;
- $F$  is nondecreasing;
- $F$  is continuous.

## 3.1 Density mass function and cdf

The probability that  $X$  lies in the interval  $[a, b]$  is computed in terms of its cdf as

$$P(a \leq X \leq b) = F_X(b) - F_X(a).$$

### Relationship between density mass function and cdf

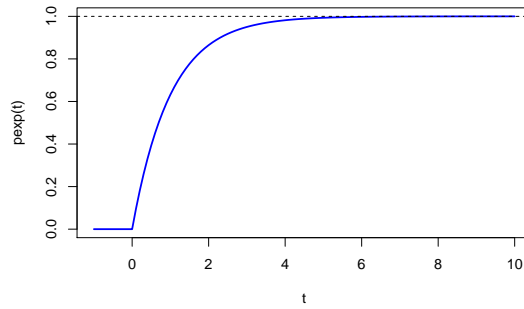
- The cdf is a primitive of the density mass function,  $F_X(x) = \int_{-\infty}^x f_X(t) dt$ .
- The density mass function is the derivative of the cdf,  $f_X(x) = F'_X(x)$ .

## Example (cdf)

$$F_X(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$P(2 < X < 4) = F_X(4) - F_X(2) = (1 - e^{-4}) - (1 - e^{-2}) = 0.117$$

```
t=seq(-1,10,by=.1)
plot(t,pexp(t),type="l",col="blue",lwd=2)
abline(h=1,lty=2)
```



## 3.2 Mean, variance, and quantiles

### Mean or expectation

The **mean** or **expectation** of  $X$  is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x) dx, .$$

### Properties of the mean

For any real numbers  $a, b \in \mathbb{R}$ , any function  $g : \mathbb{R} \mapsto \mathbb{R}$ , and r.v.  $X$ ,

- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ ;
- $\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$ ;
- $\mathbb{E}[(X - \mathbb{E}[X])^2] = \min_{x \in \mathbb{R}} \mathbb{E}[(X - x)^2]$ .

## 3.2 Mean, variance, and quantiles

### Variance

The **variance** is a measure of the *scatter* of the distribution of r.v.  $X$ .

It is the expected squared distance of  $X$  to its mean,

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{+\infty} (x - \mathbb{E}[X])^2 f_X(x) d(x).$$

### Properties of the variance

- $\text{Var}[X] \geq 0$ ;
- $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ ;
- $\text{Var}[aX + b] = a^2 \text{Var}[X]$ , for any  $a, b \in \mathbb{R}$ .

## Standard deviation

The **standard deviation** of  $X$  is the (positive) square root of its variance,

$$\sigma_X = \sqrt{\text{Var}[X]}.$$

## Example (mean, variance)

$X$  with the previous density.

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^{+\infty} x e^{-x} dx = [-x e^{-x}]_0^{+\infty} + \int_0^{+\infty} e^{-x} dx = 1.$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_0^{+\infty} x^2 e^{-x} dx = 2 \int_0^{+\infty} x e^{-x} dx = 2.$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1.$$

```
set.seed(1)
x=rexp(10000)
mean(x)
```

```
## [1] 0.9983612
```

```
var(x)
```

```
## [1] 1.031541
```

## 3.2 Mean, variance, and quantiles

### Median

The **median** is the most central value with respect to the distribution of a random variable  $X$  in the sense that

$$F_X(\text{Me}_X) = P(X \leq \text{Me}_X) = 1/2.$$

### Example

Solve  $F_X(\text{Me}_X) = 1/2$ , then  $1 - e^{-\text{Me}_X} = 1/2$ , and  $\text{Me}_X = -\log(1/2) = \log(2) = 0.693$ .

### Properties of the median

- $\text{Me}_{aX+b} = a\text{Me}_X + b$ , for any  $a, b \in \mathbb{R}$ ;
- $\text{Me}_{g(X)} = g(\text{Me}_X)$  if  $g$  is monotone;
- $\mathbb{E}|X - \text{Me}_X| = \min_{x \in \mathbb{R}} \mathbb{E}|X - x|$ .

## 3.2 Mean, variance, and quantiles

### Quantiles

For  $0 < \alpha < 1$  the  $\alpha$ -**quantile** of random variable  $X$  a number  $q_\alpha$  such that

$$F_X(q_\alpha) = P(X \leq q_\alpha) = \alpha.$$

The **quantile function** of random variable  $X$  is defined as

$$F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}.$$

A quantile function defined like this is:

- $\lim_{\alpha \downarrow 0} F_X^{-1}(\alpha) = \inf X(S)$ ;
- $\lim_{\alpha \uparrow 1} F_X^{-1}(\alpha) = \sup X(S)$ ;
- *nondecreasing*;
- *left-continuous*.

## Example (quantiles)

$X$  with the previous density. If  $F_X(x) = 1 - e^{-x} = y$ , then  $y = -\log(1 - x)$ , so

$$F_X^{-1}(y) = -\log(1 - y).$$

Half of the random variables with the distribution of  $X$  assume a value greater (or less) than  $\text{Me}_X = 0.693$ , while 75% assume a value greater than  $F^{-1}(0.25) = -\log(0.75) = 0.288$ .

```
median(x)
```

```
## [1] 0.6946537
```

```
quantile(x,0.25)
```

```
##      25%
```

```
## 0.2810167
```

### 3.3 Uniform distribution `unif(min=0,max=1)`

A **Uniform** random variable in the interval  $[a, b]$  represents a number at random in that interval selected in such a way that the probability that it lies in any subinterval of  $[a, b]$  is proportional to the width of the subinterval.

$$X \sim U(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}.$$

```
dunif(x,min=a,max=b)
```

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}.$$

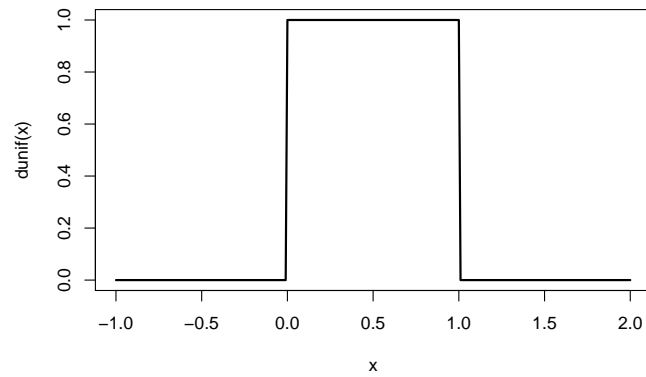
```
punif(x,min=a,max=b)
```

$$\mathbb{E}[X] = \frac{a+b}{2} \quad ; \quad \text{Var}[X] = \frac{(b-a)^2}{12}$$

### Uniform density mass function `dunif(min=0,max=1)`

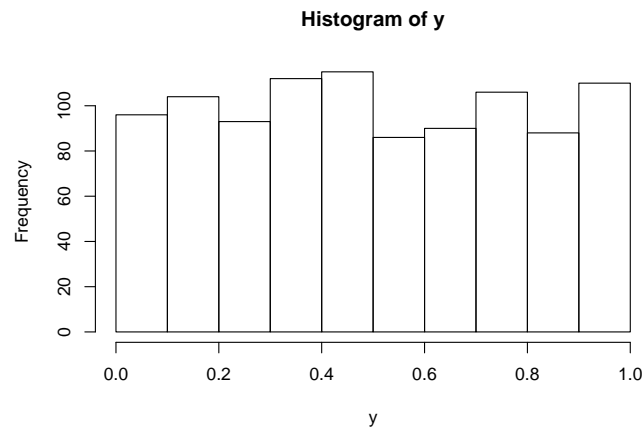
```
x=seq(-1,2,by=.01)
```

```
plot(x,dunif(x),type="l",lwd=2)
```



## Uniform random observations `runif(min=0,max=1)`

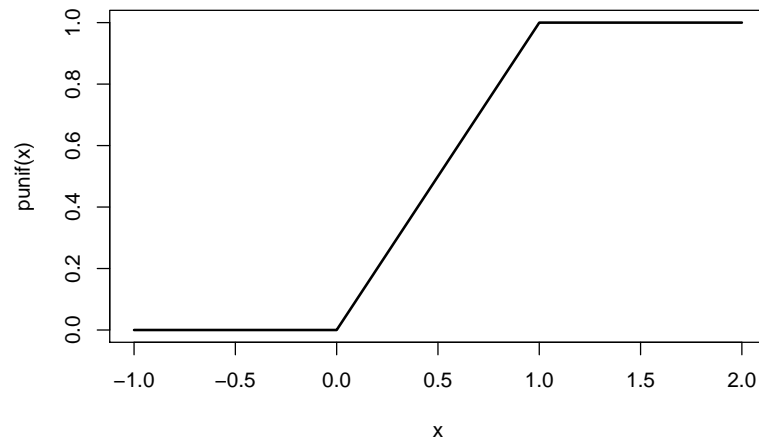
```
set.seed(1)
y=runif(1000)
hist(y)
```



## Uniform cdf `punif(min=0,max=1)`

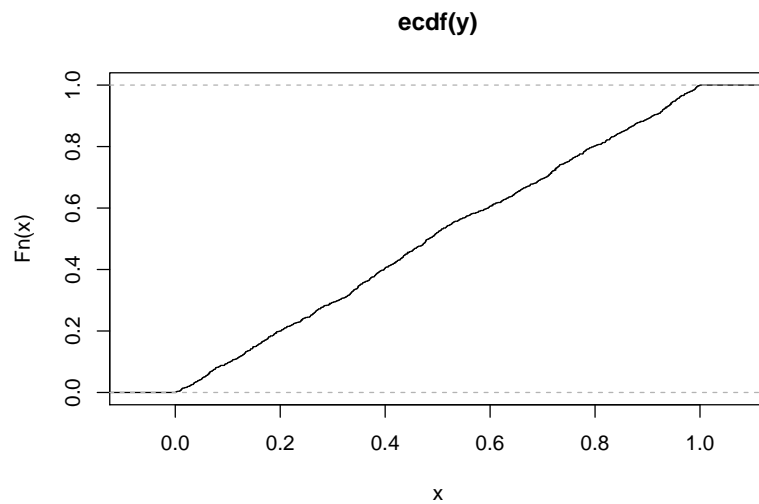
```
x=seq(-1,2,by=.01)
plot(x,punif(x),type="l",lwd=2)
```





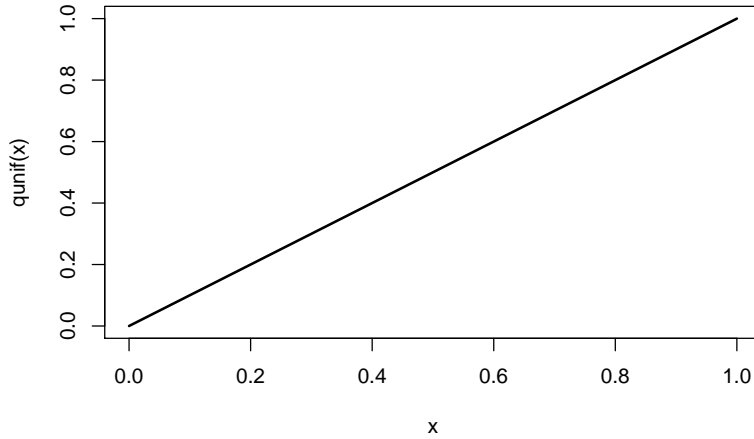
## Uniform empirical cumulative distribution function

```
plot(ecdf(y))
```



## Uniform quantile function `qunif(min=0,max=1)`

```
x=seq(0,1,by=.01)
plot(x,qunif(x),type="l",lwd=2)
```



### 3.4 Transformations of a random variable

If  $X$  is a random variable and  $g : \mathbb{R} \mapsto \mathbb{R}$  a function, then  $Y = g(X)$  is a random variable.

If  $X$  is continuous and  $g$  continuous and increasing

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)),$$

where  $g^{-1}$  is the inverse function of  $g$ , that is,  $g^{-1}(y) = x$  if  $g(x) = y$ .

In general, if  $g$  is injective (one-to-one) and derivable

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|.$$

### 3.4 Transformations of a random variable

#### Example

Consider  $X \sim U(0, 1)$ , determine the density mass function of  $Y = -\log(1 - X)$ .

Clearly the support of  $Y$  is  $(0, +\infty)$ , consider  $y > 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(-\log(1 - X) \leq y) = P(\log(1 - X) \geq -y) \\ &= P(1 - X \geq e^{-y}) = P(-X \geq e^{-y} - 1) = P(X \leq 1 - e^{-y}) = 1 - e^{-y}. \end{aligned}$$

$$F_Y(y) = \begin{cases} 1 - e^{-y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}.$$

## Inverse transform method for simulation

If  $X \sim U(0, 1)$ , then  $F^{-1}(X)$  is a random variable with cdf  $F$ .

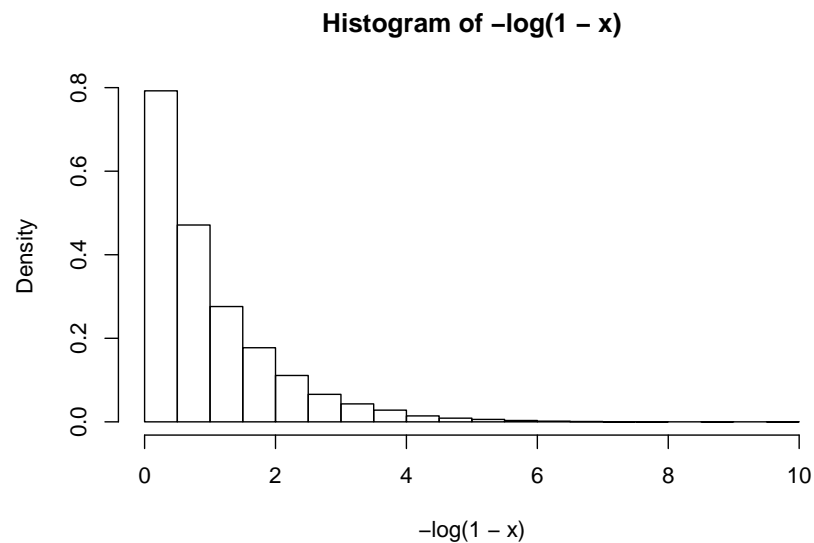
$$P(F^{-1}(X) \leq x) = P(X \leq F(x)) = F_X(F(x)) = F(x)$$

### Example

Observe that if  $F(x) = 1 - e^{-x}$  for  $x \geq 0$ , then  $F^{-1}(x) = -\log(1 - x)$ . The cdf of  $Y = -\log(1 - X)$  is  $F$  and we can use this to simulate from such a distribution.

## Inverse transform method for simulation (Example)

```
set.seed(1)
x=runif(10000)
hist(-log(1-x),probability=T)
```



## 3.5 Exponential distribution `exp(rate=1)`

If  $X \sim \mathcal{P}(\lambda)$  represents the number of events that occur in a given time period (independently and with constant rate  $\lambda$  events per time units in the period), then the time between two consecutive events follows an **Exponential** distribution with parameter  $\lambda$ .

$$\begin{aligned} X_t &\equiv \text{'number of events in } [0, t]\text{'} \\ T &\equiv \text{'time until first event occurs'} \\ X_t &\sim \mathcal{P}(\lambda t) \end{aligned}$$

Take  $t > 0$ ,

$$F_T(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(X_t = 0) = 1 - e^{-\lambda t}.$$

## Exponential distribution `exp(rate=1)`

$T \sim \text{Exp}(\lambda)$

- cdf

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

- density

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$\mathbb{E}[T] = \lambda^{-1} \quad ; \quad \text{Var}[T] = \lambda^{-2}$$

## Exponential distribution `exp(rate=1)`

### Lack of memory property

If  $T \sim \text{Exp}(\lambda)$  and  $t_1, t_2 > 0$ , then

$$P(T > t_1 + t_2 | T > t_1) = P(T > t_2).$$

*Proof:*

$$\begin{aligned} P(T > t_1 + t_2 | T > t_1) &= \frac{P((T > t_1 + t_2) \cap (T > t_1))}{P(T > t_1)} = \frac{P(T > t_1 + t_2)}{P(T > t_1)} \\ &= \frac{1 - F_T(t_1 + t_2)}{1 - F_T(t_1)} = \frac{e^{-\lambda(t_1 + t_2)}}{e^{-\lambda t_1}} = e^{-\lambda t_2} = P(T > t_2). \end{aligned}$$

## 3.6 Normal distribution `norm(mean=0, sd=1)`

Random variable  $X$  follows a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ ,  $X \sim N(\mu, \sigma)$  if its density mass function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

```
dnorm(x, mean=mu, sd=sigma)
```

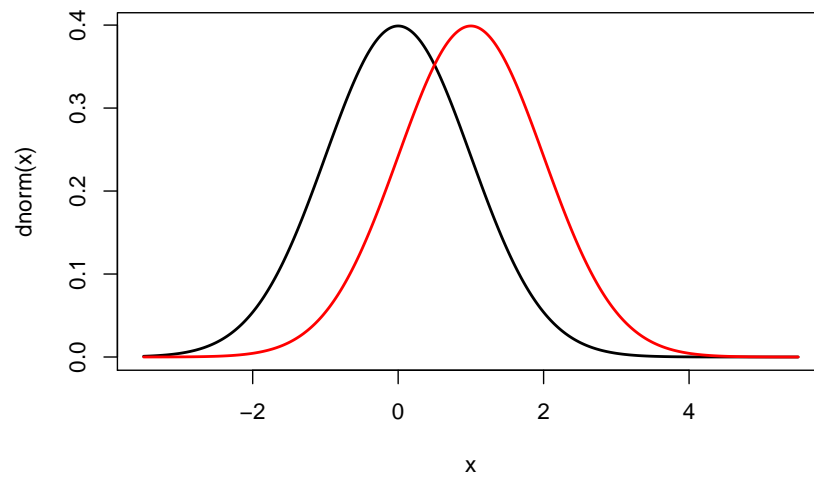
We refer to  $Z \sim N(0, 1)$  as **standard normal** random variable,

$$f_Z(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

```
dnorm(x)
```

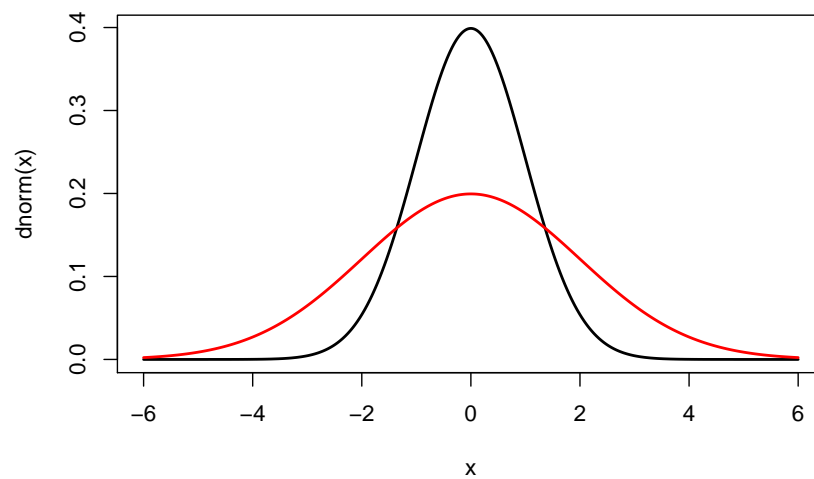
## Normal density (location shift) `dnorm(mean=0,sd=1)`

```
x=seq(-3.5,5.5,by=.01)
plot(x,dnorm(x),type="l",lwd=2)
points(x,dnorm(x,mean=1,sd=1),type="l",lwd=2,col="red")
```



## Normal density (scale shift) `dnorm(mean=0,sd=1)`

```
x=seq(-6,6,by=.01)
plot(x,dnorm(x),type="l",lwd=2)
points(x,dnorm(x,mean=0,sd=2),type="l",lwd=2,col="red")
```

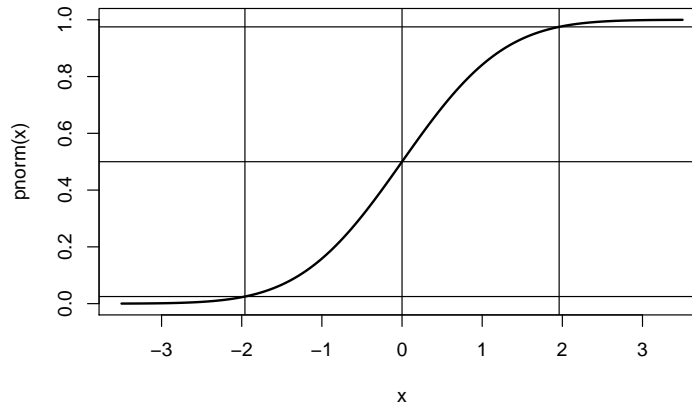


## Normal cdf `pnorm(mean=0,sd=1)`

There is no analytic expression for the cdf of a normal r.v.

If  $Z \sim N(0, 1)$ ,  $F_Z(x) = P(Z \leq z) = \int_{-\infty}^x \phi(t)dt = \Phi(x)$ .

```
x=seq(-3.5,3.5,by=.01)
plot(x,pnorm(x),type="l",lwd=2)
abline(h=c(0.025,0.5,0.975),v=c(-1.96,0,1.96))
```



## Normal distribution `norm(mean=0,sd=1)`

A linear transformation of a normal random variable is normal

If  $X \sim N(\mu, \sigma)$  and  $a, b \in \mathbb{R}$ ,

$$aX + b \sim N(a\mu + b, |a|\sigma).$$

### Standardization

Among all linear transformations of a normal r.v., the most relevant is the **standardization**, if  $X \sim N(\mu, \sigma)$ ,

$$\frac{X - \mu}{\sigma} \sim N(0, 1).$$

## Normal distribution `norm(mean=0,sd=1)`

### Examples

If  $X \sim N(\mu = 2, \sigma = 3)$ , compute:

- $P(X \leq 4)$

```
pnorm(4,mean=2,sd=3)
```

```
## [1] 0.7475075
```

- $P(X \leq 4) = P((X - 2)/3 \leq (4 - 2)/3) = \Phi(2/3)$

```
pnorm(2/3)
```

```
## [1] 0.7475075
```

## Normal approximation to the Binomial distribution

### DeMoivre-Laplace limit theorem

For  $0 < p < 1$  and  $r \in \{0, 1, 2, \dots, n\}$

$$\frac{\sqrt{2\pi np(1-p)} \binom{n}{r} p^r (1-p)^{n-r}}{e^{-(r-np)^2/(2np(1-p))}} \xrightarrow{n \rightarrow +\infty} 1$$

Consequence:

If  $X \sim B(n, p)$ , then for any  $a < b$ , we have

$$P\left(a \leq \frac{X - np}{\sqrt{np(1-p)}} \leq b\right) \xrightarrow{n \rightarrow +\infty} \Phi(b) - \Phi(a)$$

Good approximation for values of  $n$  satisfying  $np(1-p) \geq 10$ .

## Normal approximation to the Binomial distribution

$X \sim B(n = 40, p = 0.5)$

- $P(X = 20)$

```
dbinom(20, size=40, prob=0.5)
```

```
## [1] 0.1253707
```

- $P(X = 20) = P(19.5 \leq X \leq 20.5)$

```
pnorm(20.5, mean=20, sd=sqrt(10)) - pnorm(19.5, mean=20, sd=sqrt(10))
```

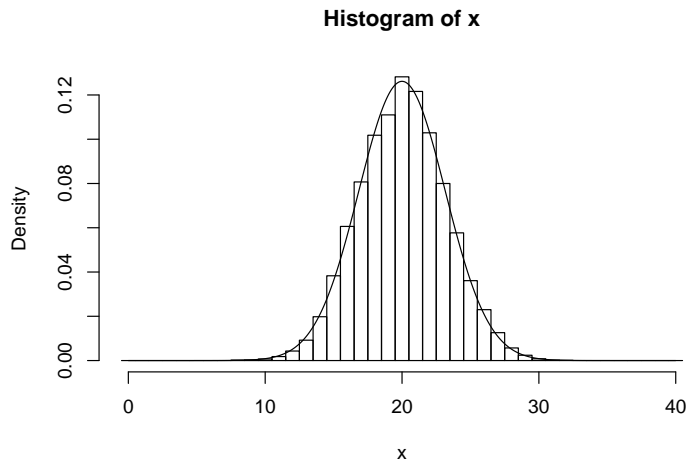
```
## [1] 0.1256329
```

```
dnorm(20, mean=20, sd=sqrt(10))
```

```
## [1] 0.1261566
```

## Normal approximation to the Binomial distribution

```
set.seed(2)
x=rbinom(10000, size=40, prob=.5)
hist(x, breaks=seq(-0.5, 40.5, 1), probability=T)
t=seq(0, 40, by=.01)
points(t, dnorm(t, mean=20, sd=sqrt(10)), type="l")
```



## Continuous distributions in R

Distributions	R command
Uniform, $U(a, b)$	<code>unif(min=0,max=1)</code>
Exponential, $\text{Exp}(\lambda)$	<code>exp(rate=1)</code>
Normal, $N(\mu, \sigma)$	<code>norm(mean=0,sd=1)</code>
Gamma, $\text{Gamma}(k, \lambda)$	<code>gamma(shape,rate=1)</code>
Beta, $\text{Beta}(\alpha, \beta)$	<code>beta(shape1,shape2)</code>
Chi-square, $\chi_n^2$	<code>chisq(df)</code>
Student's $t$ , $t_n$	<code>t(df)</code>
Fisher's $F$ , $F_{n_1, n_2}$	<code>f(df1,df2)</code>

Functions	R prefix
density	<code>d</code>
cdf	<code>p</code>
quantile function	<code>q</code>
random numbers	<code>r</code>