Properties of the expectation

Ignacio Cascos, Department of Statistics, Universidad Carlos III de Madrid

2018/19

1. Consider the five-parameter model given by the mixture of two normal distributions. Denote $X_1 \sim \mathrm{N}(\mu_1, \sigma_1)$, $X_2 \sim \mathrm{N}(\mu_2, \sigma_2)$, and X a random variable whose cdf is $F_X = pF_{X_1} + (1-p)F_{X_2}$.
   1. Determine the first four moments of X_1 and X_2.

$$\begin{aligned}
\mathbb{E}[X_i] &= \mu_i\,,\\
\mathbb{E}[X_i^2] &= \mathrm{Var}[X_i] + \mathbb{E}[X_i]^2 = \sigma_i^2 + \mu_i^2\,,\\
\mathbb{E}[X_i^3] &= \mathbb{E}[(\mu_i + \sigma_i Z)^3] = \mu_i^3 + 3\mu_i\sigma_i^2\\
\mathbb{E}[X_i^4] &= \mathbb{E}[(\mu_i + \sigma_i Z)^4] = \mu_i^4 + 6\mu_i^2\sigma_i^2 + 3\sigma_i^4
\end{aligned}$$

   2. Determine the first four moments of X.

$$\begin{aligned}
\mathbb{E}[X] &= p\mu_1 + (1-p)\mu_2\,,\\
\mathbb{E}[X^2] &= p(\sigma_1^2 + \mu_1^2) + (1-p)(\sigma_2^2 + \mu_2^2)\,,\\
\mathbb{E}[X^3] &= p(\mu_1^3 + 3\mu_1\sigma_1^2) + (1-p)(\mu_2^3 + 3\mu_2\sigma_2^2)\\
\mathbb{E}[X^4] &= p(\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4) + (1-p)(\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4)
\end{aligned}$$

   3. What are the values of the first four moments of X if $\mu_1 = 2$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = 2$, and $p = 0.4$? Simulate $n = 10000$ observations from that distribution model and check your results (use your DNI or NIU as seed for your simulations).

$$\begin{aligned}
\mathbb{E}[X] &= p\mu_1 + (1-p)\mu_2 = 2.6\,,\\
\mathbb{E}[X^2] &= p(\sigma_1^2 + \mu_1^2) + (1-p)(\sigma_2^2 + \mu_2^2) = 9.8\,,\\
\mathbb{E}[X^3] &= p(\mu_1^3 + 3\mu_1\sigma_1^2) + (1-p)(\mu_2^3 + 3\mu_2\sigma_2^2) = 43.4\\
\mathbb{E}[X^4] &= p(\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4) + (1-p)(\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4) = 224.2
\end{aligned}$$

```r
set.seed(1)
arguments=sample(1:2,prob=c(0.4,0.6),size=10000,replace=T)
mus=c(2,3); sds=c(1,2)
x=rnorm(10000,mean=mus[arguments],sd=sds[arguments])
mean(x)
```

```
## [1] 2.581456
```

```r
mean(x^2)
```

```
## [1] 9.666662
```

```r
mean(x^3)
```

```
## [1] 42.65037
```

```r
mean(x^4)
```

```
## [1] 219.8188
```

2. Claims arrive in an insurance company with regard to a Poisson process, that is, the number of claims received in a time period is a Poisson random variable N. The size of each individual claim is a random variable X_i that models its cost. The X_is are independent and independent of N. Assume that each individual claim X_i follows a log-normal distribution with parameters μ and σ, that is, $\log(X_i) \sim \text{N}(\mu, \sigma)$ and the mean and variance of this distribution model are

$$\mathbb{E}[X_i] = e^{\mu+\sigma^2/2} \quad \text{and} \quad \text{Var}[X_i] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}.$$

The total expediture that the insurance company must carry out to cover the claims is

$$Y = \sum_{i=1}^{N} X_i\,.$$

1. Determine $\mathbb{E}[Y]$ and $\text{Var}[Y]$.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[N]\mathbb{E}[X] = \lambda e^{\mu+\sigma^2/2}\\ \text{Var}[Y] &= \mathbb{E}[N]\text{Var}[X] + \mathbb{E}[X]^2\text{Var}[N] = \lambda(e^{\sigma^2} - 1)e^{2\mu+\sigma^2} + \lambda e^{2\mu+\sigma^2} = \lambda e^{2\sigma^2+2\mu}.\end{aligned}$$

2. What values do $\mathbb{E}[Y]$ and $\text{Var}[Y]$ assume if $\lambda = 100$, $\mu = 6$, and $\sigma = 2.5$?

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[N]\mathbb{E}[X] = \lambda e^{\mu+\sigma^2/2} = 9.181997 \times 10^5\\ \text{Var}[Y] &= \mathbb{E}[N]\text{Var}[X] + \mathbb{E}[X]^2\text{Var}[N] = \lambda e^{2\sigma^2+2\mu} = 4.3673179 \times 10^{12}.\end{aligned}$$

3. Simulate $n = 10000$ observations of Y in order to check your results (use your DNI or NIU as seed for your simulations).

```r
set.seed(2)
x=vector(length=10000)
for(i in 1:10000){
  N=rpois(1,lambda=100)
  x[i]=sum(exp(rnorm(N,mean=6,sd=2.5)))
}
mean(x)
```

```
## [1] 951528
```

```r
var(x)
```

```
## [1] 4.336184e+12
```

4. Assume now that the size of each individual claim, X_i, follows a mixture of two log-normal distributions with probability 1/2 each and respective parameters μ_1 and σ_1, and μ_2 and σ_2. Determine $\mathbb{E}[Y]$ and $\text{Var}[Y]$.

$$\begin{aligned}\mathbb{E}[X] &= (\mathbb{E}[X_1] + \mathbb{E}[X_2])/2 = (e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2})/2\\ \text{Var}[X] &= (\mathbb{E}[X_1^2] + \mathbb{E}[X_1^2])/2 - \mathbb{E}[X]^2 = (e^{2\mu_1+2\sigma_1^2} + e^{2\mu_2+2\sigma_2^2})/2 - [(e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2})/2]^2\\ \mathbb{E}[Y] &= \mathbb{E}[N]\mathbb{E}[X] = (\lambda/2)(e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2})\\ \text{Var}[Y] &= \mathbb{E}[N]\text{Var}[X] + \mathbb{E}[X]^2\text{Var}[N]\\ &= \lambda\left((e^{2\mu_1+2\sigma_1^2} + e^{2\mu_2+2\sigma_2^2})/2 - [(e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2})/2]^2\right) + \lambda[(e^{\mu_1+\sigma_1^2/2} + e^{\mu_2+\sigma_2^2/2})/2]^2\\ &= (\lambda/2)(e^{2\mu_1+2\sigma_1^2} + e^{2\mu_2+2\sigma_2^2}).\end{aligned}$$