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4. Random vectors (1/2)

Problem 1. Given the discrete random vector (X, Y) with the following probability mass function

$$p_{X,Y}(x, y) = \begin{cases} c(x^2 + y^2) & \text{si } x \in \{1, 2, 4\},\ y \in \{1, 3\} \\ 0 & \text{otherwise} \end{cases} .$$

a) Compute the value of c.

b) Compute the probabilities: $P(Y < X)$, $P(Y > X)$, and $P(Y = X)$

c) Compute $P(Y = 3)$

d) Compute the marginal probability mass function $p_X(x)$ and $\mathbb{E}[X]$.

Problem 2. A dartboard is a wooden square of 40cm × 40cm. Inside the board there are several concentric circles, the greater among them having a diameter of 40cm. The darts of an unskilled player hit the board at a point that can be described by a random vector (X, Y) with joint density function

$$f_{X,Y}(x, y) = \begin{cases} 1/1600 & \text{if} & 0 \le x, y \le 40 \\ 0 & \text{otherwise} & \end{cases} ,$$

where (X, Y) represent the coordinates (in cm) of the hitting point. The lower left corner of the board is represented by the origin, that is, the point (0,0).

a) Whenever we hit one of the circles, we get some point. What is the probability of getting some point from a throw?

b) What is the mean vector of (X, Y)?, and the probability that (X, Y) assumes exactly that value?

c) The inner circle has a diameter of 4cm. What is the probability of hitting it with a dart?

Hint: If $A \subset \mathbb{R}^2$ and $k \in \mathbb{R}$, then $\int\int_A k\,dx\,dy = k\text{Area}(A)$.

Problem 3. Given the random variables X, Y, Z satisfying

$$X = 3Y + 2Z, \qquad \text{Var}[Y] = \text{Var}[Z] = 1, \qquad \mathbb{E}[Y] = 2,$$

$$\text{Cov}[Y, Z] = -0.5, \qquad \mathbb{E}[Z] = -3$$

Compute the following:

a) $\mathbb{E}[X]$

b) $\text{Var}[X]$

c) $\text{Cov}[X, Y]$

d) $\rho_{X,Y}$, where ρ stands for correlation

Problem 4. Data frame `cars` contains information on two numeric variables: `speed` (mph) and `dist` (feet).

1. How many observations does `cars` contain?
2. What are the means of `speed` and `dist`?
3. What are the variances of `speed` and `dist`?
4. What are the covariance and correlation of `speed` and `dist`?
5. Assume the random vector (`speed`, `dist`) follows a bivariate normal distribution with mean as computed in 2. and covariance matrix as in 3. and 4. Compute (a) the probability that the distance to stop is greater than 80 feet, and (b) the probability that the distance minus twice the speed is less than 20.
6. Transform the speed to kilometers per hour and the distance to meters. What are the new mean vector and covariance matrix?

Problem 5. Four flatmates Alice, Bob, Charly, and Dave roll a 6-sided die every night in order to decide who washes the dishes after dinner. If the outcome is 1, Alice washes, it is 2, Bob does, while for 3 or 4 Charly must wash the dishes and for 5 or 6 it is Dave's turn.

1. What is the probability that Alice and Bob wash the dishes once in a week (7 days) each, and Charly twice?
2. What is the probability that Alice and Bob wash the dishes once in a week each?