Bayesian Analysis of Stochastic Process Models

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POISSON PROCESS

- One of the simplest and most applied stochastic processes
- Used to model occurrences (and counts) of rare events in time and/or space, when they are not affected by past history
- Applied to describe and forecast incoming telephone calls at a switchboard, arrival of customers for service at a counter, occurrence of accidents at a given place, visits to a website, earthquake occurrences and machine failures, to name but a few applications
- Special case of CTMCs with jumps possible only to the next higher state and pure birth processes, as well as model for arrival process in $M/G/c$ queueing systems
- Simple mathematical formulation and relatively straightforward statistical analysis $\Rightarrow$ very practical, if approximate, model for describing and forecasting many random events

POISSON PROCESS

- Counting process $N(t), t \geq 0$: stochastic process counting number of events occurred up to time t

- $N(s,t]$, $s < t$: number of events occurred in time interval $(s,t]$

- Poisson process with intensity function $\lambda(t)$: counting process $N(t), t \geq 0$, s.t.
  1. $N(0) = 0$
  2. Independent number of events in non-overlapping intervals
  3. $P\left(N(t, t + \Delta t] = 1\right) = \lambda(t)\Delta t + o(\Delta t)$, as $\Delta t \to 0$
  4. $P\left(N(t, t + \Delta t] \geq 2\right) = o(\Delta t)$, as $\Delta t \to 0$

- Definition $\Rightarrow P(N(s,t] = n) = \dfrac{(\int_s^t \lambda(x)dx)^n}{n!} e^{-\int_s^t \lambda(x)dx}$, for $n \in \mathbb{Z}^+$
  $\Rightarrow N(s,t] \sim \text{Po}\left(\int_s^t \lambda(x)dx\right)$

POISSON PROCESS

- Intensity function: $\lambda(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t] \geq 1)}{\Delta t}$
  - HPP (homogeneous Poisson process): constant $\lambda(t) = \lambda$, $\forall t$
  - NHPP (nonhomogeneous Poisson process): o.w.
- HPP with rate λ
  - $N(s, t] \sim \text{Po}(\lambda(t - s))$
  - Stationary increments (distribution dependent only on interval length)

POISSON PROCESS

- Mean value function $m(t) = E[N(t)], t \geq 0$
- $m(s,t] = m(t) - m(s)$ expected number of events in $(s,t]$
- If $m(t)$ differentiable, $\mu(t) = m'(t)$, $t \geq 0$, Rate of Occurrence of Failures (ROCOF)
- $P\left(N(t, t + \Delta t] \geq 2\right) = o(\Delta t)$, as $\Delta t \to 0$
  $\Rightarrow$ orderly process
  $\Rightarrow \lambda(t) = \mu(t)$ a.e.
- $\Rightarrow m(t) = \int_0^t \lambda(x)dx$ and $m(s,t] = \int_s^t \lambda(x)dx$
- $\Rightarrow m(t) = \lambda t$ and $m(s,t] = \lambda(t - s)$ for HPP with rate λ

POISSON PROCESS

- Arrival times $\{T_n, n \in \mathbb{Z}^+\}$:

$$T_n := \begin{cases} \min\{t : N(t) \geq n\} & n > 0 \\ 0 & n = 0 \end{cases}$$

- $\{T_n, n \in \mathbb{N}^+\}$ stochastic process, sort of dual of $N(t)$

- Interarrival times $\{X_n, n \in \mathbb{Z}^+\}$:

$$X_n := \begin{cases} T_n - T_{n-1} & n > 0 \\ 0 & n = 0 \end{cases}$$

- Interarrival and arrival times related through $T_n = \sum_{i=1}^{n} X_i$

POISSON PROCESS

- In a Poisson process $N(t)$ with intensity $\lambda(t)$
  - T_n has density $g_n(t) = \dfrac{\lambda(t)[m(t)]^{n-1}}{\Gamma(n)} e^{-m(t)}$
  - X_n has distribution function, conditional upon the occurrence of the $(n-1)$-st event at T_{n-1}, given by $F_n(x) = \dfrac{F(T_{n-1}+x) - F(T_{n-1})}{1 - F(T_{n-1})}$, with $F(x) = 1 - e^{-m(x)}$ (or $F_n(x) = 1 - \exp\{-[m(T_{n-1}+x) - m(T_{n-1})]\}$)
  - $\Rightarrow$ distribution function of T_1 (and X_1) given by $F_1(t) = 1 - e^{-m(t)}$, and density function $g_1(t) = \lambda(t)e^{-m(t)}$
- For HPP with rate λ
  - Interarrival times, and first arrival time, have an exponential distribution Ex(λ) ($\Rightarrow$ HPP renewal process)
  - n-th arrival time, T_n, has a gamma distribution Ga(n, λ), for each $n \geq 1$
  - Link between HPP and exponential distribution

POISSON PROCESS

Poisson process $N(t)$ with intensity function $\lambda(t)$ and mean value function $m(t)$

- $T_1 < \ldots < T_n$: n arrival times in $(0, T] \Rightarrow P(T_1, \ldots, T_n) = \prod_{i=1}^{n} \lambda(T_i) \cdot e^{-m(T)}$

  $\Rightarrow$ likelihood

- $\Rightarrow P(T_1, \ldots, T_n) = \lambda^n e^{-\lambda T}$ for HPP with rate λ

- n events occur up to time $t_0 \Rightarrow$ distributed as order statistics from cdf $m(t)/m(t_0)$, for $0 \leq t \leq t_0$ (uniform distribution for HPP)

POISSON PROCESS

- Under suitable conditions, Poisson processes can be merged or split to obtain new Poisson processes (see Kingman, p. 14 and 53, 1993)

- Useful in applications, e.g.
  - merging gas escapes from pipelines installed in different periods
  - splitting earthquake occurrences into minor and major ones

- **Superposition Theorem**
  - n independent Poisson processes $N_i(t)$, with intensity function $\lambda_i(t)$ and mean value function $m_i(t)$, $i = 1, \ldots, n$
  - $\Rightarrow N(t) = \sum_{i=1}^{n} N_i(t)$, for $t \geq 0$, Poisson process with intensity function $\lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$ and mean value function $m(t) = \sum_{i=1}^{n} m_i(t)$

POISSON PROCESS

- **Coloring Theorem**
  - $N(t)$ be a Poisson process with intensity function $\lambda(t)$
  - Multinomial random variable Y, independent from the process, taking values $1, \ldots, n$ with probabilities $p_1, \ldots, p_n$
  - Each event assigned to classes (*colors*) $A_1, \ldots, A_n$ according to $Y \Rightarrow n$ independent Poisson processes $N_1(t), \ldots, N_n(t)$ with intensity functions $\lambda_i(t) = p_i \lambda(t)$, $i = 1, \ldots, n$

- Coloring Theorem extended to the case of time dependent probabilities $p(t)$, defined on $(0, \infty)$
  - As an example, for an HPP with rate λ, if events at any time t are kept with probability $p(t) \Rightarrow$ Poisson process with intensity function $\lambda p(t)$

POISSON PROCESS: INFERENCE

- $N(t)$ HPP with parameter λ
- n events observed in the interval $(0, T]$
- Likelihood for two possible experiments
  - Times $T_1, \ldots, T_n$ available

    Theorem on Poisson processes $\Rightarrow l(\lambda|data) = \lambda^n e^{-\lambda T}$
  - Only number n available

    Properties of Po$(\lambda T) \Rightarrow l(\lambda|data) = \dfrac{(\lambda T)^n}{n!} e^{-\lambda T}$
- Proportional likelihoods $\Rightarrow$ same inferences (Likelihood Principle, Berger and Wolpert, 1988)
- In both cases, likelihood not dependent on the actual occurrence times but only on their number

POISSON PROCESS: INFERENCE

- Gamma priors conjugate w.r.t. λ in the HPP
- Prior Ga(α, β)
- $\Rightarrow f(\lambda|n, T) \propto \lambda^n e^{-\lambda T} \cdot \lambda^{\alpha-1} e^{-\beta\lambda}$
- $\Rightarrow$ posterior Ga$(\alpha + n, \beta + T)$
- Posterior mean $\widehat{\lambda} = \dfrac{\alpha + n}{\beta + T}$
- Posterior mean combination of
  - Prior mean $\widehat{\lambda}_P = \dfrac{\alpha}{\beta}$
  - MLE $\widehat{\lambda}_M = \dfrac{n}{T}$

ACCIDENTS IN THE CONSTRUCTION SECTOR

Ríos Insua et al (1999)

- Interest in number of accidents in some companies in the Spanish construction sector
- 75 accidents and an average number of workers of 364 in 1987 for one company
- Number of workers constant during the year
- Times of all accidents of each worker are recorded
- Accidents occur *randomly* $\Rightarrow$ HPP model justified
- Each worker has the same propensity to have accidents $\Rightarrow$
  - HPP with same λ for all of them
  - If one year corresponds to $T = 1 \Rightarrow$ number of accidents for each worker follows the same Poisson distribution Po(λ)

ACCIDENTS IN THE CONSTRUCTION SECTOR

- Accidents of different workers are independent
  - Apply Superposition Theorem
  - $\Rightarrow$ Number of accidents for all workers given by an HPP with rate 364λ
- Gamma prior Ga$(1,1)$ on λ
  - Likelihood $l(\lambda|data) = (364\lambda)^{75} e^{-364\lambda}$
  - Posterior gamma Ga$(76,365)$
  - Posterior mean $76/365 = 0.208$
  - Prior mean 1
  - MLE $75/364 = 0.206$
  - Posterior mean closer to MLE $\Rightarrow$ think of the hypothetical experiment (just 1 hypothetical sample w.r.t. 75 actual ones)

ACCIDENTS IN THE CONSTRUCTION SECTOR

- Prior $\mathrm{Ga}(1, 1) \Rightarrow$ mean 1 and variance 1
  - *large* variance in this experiment
  - $\Rightarrow$ scarce confidence on the prior assessment of mean equal to 1
- Prior $\mathrm{Ga}(1000, 1000) \Rightarrow$ mean 1 and variance 0.001
  - *Small* variance in this experiment
  - $\Rightarrow$ strong confidence on the prior assessment of mean equal to 1
- $\Rightarrow$ Posterior $\mathrm{Ga}(1075, 1364)$
- Posterior mean $1075/1364 = 0.79$
- Prior mean 1
- MLE $75/364 = 0.21$
- Posterior mean $10075/10364 = 0.97$ for a $\mathrm{Ga}(10000, 10000)$ prior

POISSON PROCESS: INFERENCE

- Computation of quantities of interest
  - analytically (e.g. posterior mean and mode)
  - using basic statistical software (e.g. posterior median and credible intervals)

- Accidents in the construction sector
  - Gamma prior Ga$(100, 100)$ for the rate λ
  - Posterior mean: $175/464 = 0.377$
  - Posterior mode: $174/464 = 0.375$
  - Posterior median: 0.376
  - $[0.323, 0.435]$: 95% credible interval $\Rightarrow$ quite concentrated distribution
  - Posterior probability of interval $[0.3, 0.4]$: 0.789

NON CONJUGATE ANALYSIS

- Improper priors
  - Controversial, although rather common, choice, which might reflect lack of knowledge
  - Possible choices
    * $f(\lambda) \propto 1$: Uniform prior
      $\Rightarrow$ posterior Ga$(n+1, T)$
    * $f(\lambda) \propto 1/\lambda$: Jeffreys prior given the experiment of observing times between events
      $\Rightarrow$ posterior Ga$(n/2, T)$
    * $f(\lambda) \propto 1/\sqrt{\lambda}$: Jeffreys prior given the experiment of observing the number of events in a fixed period
      $\Rightarrow$ posterior Ga$(n+1/2, T)$

NON CONJUGATE ANALYSIS

- Lognormal prior LN$\big(\mu, \sigma^2\big)$
- $\Rightarrow$ posterior $f(\lambda|n,T) \propto \lambda^n e^{-\lambda T} \cdot \lambda^{-1} e^{-(\log\lambda - \mu)/(2\sigma^2)}$
- Normalizing constant C and other quantities of interest (e.g., the posterior mean $\widehat{\lambda}$) computed numerically, using, e.g., Monte Carlo simulation

1. `Set` $C = 0$, $D = 0$ `and` $\widehat{\lambda} = 0$. $i = 1$.
2. `While` $i \leq M$, `iterate through`
   - `Generate` λ_i `from a lognormal distribution LN`$\big(\mu, \sigma^2\big)$
   - `Compute` $C = C + \lambda_i^n e^{-\lambda_i T}$
   - `Compute` $D = D + \lambda_i^{n+1} e^{-\lambda_i T}$
   - $i = i + 1$
3. `Compute` $\widehat{\lambda} = \frac{D}{C}$.

NON CONJUGATE ANALYSIS

- Given the meaning of λ (expected number of events in unit time interval or inverse of mean interarrival time), it may often be considered that λ is bounded

- $\Rightarrow$ Prior on a bounded set

- Uniform prior on the interval $(0, L]$

- $\Rightarrow$ Posterior $f(\lambda|n, T) \propto \lambda^n e^{-\lambda T} I_{(0,L]}(\lambda)$

- Normalizing constant $\gamma(n+1, LT)/T^{n+1}$, with $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ lower incomplete gamma function

- Posterior mean $\hat{\lambda} = \frac{1}{T}\frac{\gamma(n+2, LT)}{\gamma(n+1, LT)}$

FORECASTING

- n events observed in the interval $(0, T]$
- Interest in forecasting number of events in subsequent intervals: $P(N(T, T+s] = m)$,
- For $s > 0$ and integer m

$$\begin{aligned} P(N(T, T+s] = m) &= \int_0^\infty P(N(T, T+s] = m|\lambda)\, f(\lambda|n, T)\, d\lambda \\ &= \int_0^\infty \frac{(\lambda s)^m}{m!} e^{-\lambda s}\, f(\lambda|n, T)\, d\lambda \end{aligned}$$

Posterior Ga$(\alpha + n, \beta + T)$

$$\Rightarrow P(N(T, T+s] = m) = \frac{s^m}{m!} \frac{(\beta + T)^{\alpha+n}}{(\beta + T + s)^{\alpha+n+m}} \frac{\Gamma(\alpha + n + m)}{\Gamma(\alpha + n)}$$

FORECASTING

- Expected number of events in the subsequent interval

$$
\begin{aligned}
E[N(T,T+s]] &= \int_0^\infty E[N(T,T+s]|\lambda] f(\lambda|n,T)d\lambda \\
&= \int_0^\infty \lambda s\, f(\lambda|n,T)d\lambda
\end{aligned}
$$

Posterior Ga$(\alpha+n,\beta+T)$

$\Rightarrow E[N(T,T+s]] = s\dfrac{\alpha+n}{\beta+T}$

ACCIDENTS IN THE CONSTRUCTION SECTOR

- Gamma prior Ga$(100, 100)$ for the rate λ
- Posterior gamma Ga$(175, 464)$, having observed 75 accidents with 346 workers in 1987
- Interest in number of accidents during the first six months of 1988 (i.e. $s = 0.5$), when the number of workers has increased to 400 (i.e. $m = 400$)
- T_{1987} denotes December, 31st, 1987
- $\Rightarrow N(T_{1987}, T_{1987} + 0.5] \sim \text{Po}(400\lambda \cdot 0.5)$
- $E[N(T_{1987}, T_{1987} + 0.5]] = 400 \cdot 0.5 \dfrac{175}{464} = 75.431$
- Interested in probability of 100 accidents in the six months:
  $$P(N(T_{1987}, T_{1987} + 0.5] = 100) = \frac{200^{100}}{100!} \frac{464^{175}}{664^{275}} \frac{\Gamma(275)}{\Gamma(175)} = 0.003$$
- Probability of no accidents in the six months: $(464/664)^{175} \approx 0$

CONCOMITANT POISSON PROCESSES

- k Poisson processes $N_i(t)$, with parameter λ_i, $i = 1, \ldots, k$
- Observe n_i events over an interval $(0, t_i]$ for each process $N_i(t)$
- Processes could be related at different extent
- Typical example, as in Cagno et al (1999), provided by gas escapes in a network of pipelines which might differ in, e.g., location and environment
- Based on such features, pipelines could be split into subnetworks and an HPP for gas escapes in each of them is considered
- Some possible mathematical relations among the HPPs are presented, using the gas escape example for illustrative purposes

CONCOMITANT POISSON PROCESSES

- **Independence**
  - Processes correspond to completely different phenomena, e.g., gas escapes in completely different pipelines, for material, location, environment, etc.
  - Completely different λ_i, with no relation
  - Conjugate gamma priors Ga(α_i, β_i) for each process $N_i(t)$, $i = 1, \ldots, k$
  - $\Rightarrow$ Posterior distribution Ga$(\alpha_i + n_i, \beta_i + t_i)$
- **Complete similarity**
  - Identical processes with same λ, e.g., the gas pipelines are identical for material, laying procedure, environment and operation
  - Likelihood $l(\lambda|data) \propto \displaystyle\prod_{i=1}^{k} (\lambda t_i)^{n_i} e^{-\lambda t_i} \propto \lambda^{\sum_{i=1}^k n_i} e^{-\lambda \sum_{i=1}^k t_i}$
  - Gamma prior Ga(α, β)
  - $\Rightarrow$ Gamma posterior Ga$\left(\alpha + \sum_{i=1}^k n_i, \beta + \sum_{i=1}^k t_i\right)$

CONCOMITANT POISSON PROCESSES

Partial similarity (exchangeability):

- Processes with similar λ_i, i.e. different but from same prior, corresponding e.g. to similar, but not identical, conditions for the gas pipelines

- Hierarchical model:

$$\begin{aligned} N_i(t_i)|\lambda_i &\sim \text{Po}(\lambda_i t_i),\ i = 1, \ldots, k \\ \lambda_i|\alpha,\beta &\sim \text{Ga}(\alpha,\beta),\ i = 1, \ldots, k \\ f(\alpha,\beta) & \end{aligned}$$

- Experiment corresponding to observe k HPPs $N_i(t)$, $i = 1, \ldots, k$, with parameter λ_i, until time t_i

- Different choices for prior $f(\alpha,\beta)$ proposed in literature

CONCOMITANT POISSON PROCESSES

Different choices for prior $f(\alpha, \beta)$

- Albert (1985)
  - Model reparametrization
    - $\mu = \alpha/\beta$
    - $\gamma_i = \beta/(t_i + \beta), i = 1, \ldots, k$
  - Noninformative priors $f(\mu) = 1/\mu$ and $f(\gamma) = \gamma^{-1}(1-\gamma)^{-1}$
  - Approximations to estimate mean and variance of λ_i's
- George et al (1993): failures of ten power plants
  - Exponential priors Ex(1) or Ex(0.01) for α
  - Gamma priors Ga$(0.1, 1)$ and Ga$(0.1, 0.01)$ for β
  - Informal sensitivity analysis, with *small* and *large* values of α and β

CONCOMITANT POISSON PROCESSES

Masini et al (2006)

- Priors

$$\begin{array}{rcl} f(\alpha) & \propto & \Gamma(\alpha+1)^k/\Gamma(k\alpha+a),\ k \text{ integer } \geq 2, a > 0 \\ \beta & \sim & \mathrm{Ga}(a,b) \end{array}$$

- Proper prior distribution on α
  - Shape depends on the parameter a, appearing also in the prior on β
  - For $a > 1$, decreasing density for all positive α
  - For $a \leq 1$, increasing density up to its mode, $(1-a)/(k-1)$, and then decreasing
  - Numerical experiments showed wide range of behaviors representing possible different beliefs
  - Mathematical convenience: both gamma functions in the prior cancel when integrating out α and β and computing the posterior distribution of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$

CONCOMITANT POISSON PROCESSES

- Posterior distribution of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ given by

$$f(\boldsymbol{\lambda}|data) \propto \frac{\prod_{i=1}^k \lambda_i^{N_i(t_i)-1} \exp\{-\lambda_i t_i\}}{(\sum_{i=1}^k \lambda_i + b)^a (-\log H(\boldsymbol{\lambda}))^{k+1}},$$

with $H(\boldsymbol{\lambda}) = \prod_{i=1}^k \lambda_i (\sum_{i=1}^k \lambda_i + b)^{-k}$

- Normalizing constant C computed numerically, using, e.g., Monte Carlo simulation

1. `Set` $C = 0$. $i = 1$.
2. `Until convergence is detected, iterate through`
   - `For` $j = 1, \ldots, k$ `generate` $\lambda_j^{(i)}$ `from Ga(`$N_j(t_j), t_j$`)`
   - `Compute` $H^{(i)}(\boldsymbol{\lambda}) = \prod_{m=1}^k \lambda_m^{(i)} (\sum_{n=1}^k \lambda_n^{(i)} + b)^{-k}$
   - `Compute` $C = \sum_{l=1}^i \frac{\prod_{j=1}^k \Gamma(N_j(t_j)) t_j^{-N_j(t_j)}}{(\sum_{m=1}^k \lambda_m^{(i)} + b)^a (-\log H^{(i)}(\boldsymbol{\lambda}))^{k+1}}$
   - $i = i + 1$

COVARIATES

- Masini et al (2006): two models considering covariates
  1. directly in the parameters
  2. in the prior distributions of the parameters
- Key idea: find relations among processes through their covariates
- Two different gas subnetworks could differ on the pipe diameter (small vs. large) but they might share the location $\Rightarrow$ data from all subnetworks used to determine contribution of the covariate (diameter) in inducing gas escapes
- m covariates taking, for simplicity, values 0 or $1 \Rightarrow 2^m$ possible combinations
- For each combination $j, j = 1, \ldots, 2^m$
  - Covariate values $(X_{j1}, \ldots, X_{jm})$
  - $\Rightarrow$ Poisson process $N_j(t)$ with parameter $\lambda \prod_{i=1}^m \mu_i^{X_{ji}}$

    $\Rightarrow$ All null covariates $\Rightarrow$ HPP with parameter λ

COVARIATES

- Consider only one covariate ($m = 1$)
  - Only two possible combinations (e.g., small vs. large diameter in the gas pipelines)
  - $\Rightarrow$ Two HPPs $N_1(t)$ and $N_2(t)$ with rates λ and $\lambda\mu$, respectively
- Experiment
  - n_0 events observed in $(0, t_0]$ for 0-valued covariate
  - n_1 events observed in $(0, t_1]$ for 1-valued covariate

  $\Rightarrow$ likelihood $l(\lambda, \mu|data) \propto (\lambda t_0)^{n_0} e^{-\lambda t_0} \cdot (\lambda\mu t_1)^{n_1} e^{-\lambda\mu t_1}$
- Gamma priors Ga(α, β) and Ga(γ, δ) for λ and μ, respectively
- $\Rightarrow$ full conditional posteriors

$$\begin{aligned} \lambda|\mathbf{n}, \mathbf{t}, \mu &\sim \text{Ga}(\alpha + n_0 + n_1, \beta + t_0 + \mu t_1) \\ \mu|\mathbf{n}, \mathbf{t}, \lambda &\sim \text{Ga}(\gamma + n_1, \delta + \lambda t_1), \end{aligned}$$

  with $\mathbf{n} = (n_0, n_1)$ and $\mathbf{t} = (t_0, t_1)$

COVARIATES

- No closed forms available for posteriors

- Sample easily obtained through Gibbs sampling

  1. `Choose initial values` λ^0, μ^0. $i = 1$.
  2. `Until convergence is detected, iterate through`
     . `Generate` $\lambda^i|\mathbf{n}, \mathbf{t}, \mu^{i-1} \sim \texttt{Ga}\big(\alpha + n_0 + n_1, \beta + t_0 + \mu^{i-1}t_1\big)$
     . `Generate` $\mu^i|\mathbf{n}, \mathbf{t}, \lambda^i \sim \texttt{Ga}\big(\gamma + n_1, \delta + \lambda^i t_1\big)$
     . $i = i + 1$

- Straightforward extension to more than one covariate
  - Independent gamma priors
  - $\Rightarrow$ Full gamma conditional posteriors
  - $\Rightarrow$ Gibbs sampling

COVARIATES

- k Poisson processes $N_i(t)$ with covariates $\mathbf{X}_i' = (X_{i1}, \ldots, X_{im})$
- Covariates introduced in previous hierarchical model

$$\begin{array}{rcl} N_i(t_i)|\lambda_i & \sim & \text{Po}(\lambda_i t_i),\ i = 1, \ldots, k \\ \lambda_i|\alpha, \boldsymbol{\beta} & \sim & \text{Ga}\big(\alpha \exp\{\mathbf{X}_i'\boldsymbol{\beta}\}, \alpha\big),\ i = 1, \ldots, k \\ f(\alpha, \boldsymbol{\beta}) & & \end{array}$$

- $\Rightarrow$ $\exp\{\mathbf{X}_i'\boldsymbol{\beta}\}$ prior mean of each λ_i
- Proper priors chosen for both α and $\boldsymbol{\beta}$
- $\Rightarrow$ posterior sampled using MCMC
- Possible alternative: $\lambda_i|\alpha, \boldsymbol{\beta} \sim \text{Ga}\big(\alpha \exp\{2\mathbf{X}_i'\boldsymbol{\beta}\}, \alpha \exp\{\mathbf{X}_i'\boldsymbol{\beta}\}\big)$
- $\Rightarrow$ prior mean $\exp\{\mathbf{X}_i'\boldsymbol{\beta}\}$ and variance $1/\alpha$ (not dependent on covariates)

COVARIATES

Empirical Bayes alternative

- Fixed α
  - Perform sensitivity analysis w.r.t. α
- Estimate $\boldsymbol{\beta}$ following an empirical Bayes approach
  - Find $\widehat{\boldsymbol{\beta}}$ maximizing $P(N_1(t_1) = n_1, \ldots, N_k(t_k) = n_k|\boldsymbol{\beta}) =$

    $$= \int P(N_1(t_1) = n_1, \ldots, N_k(t_k) = n_k|\lambda_1, \ldots, \lambda_k) f(\lambda_1, \ldots, \lambda_k|\boldsymbol{\beta}) d\lambda_1 \ldots d\lambda_k$$
- $\Rightarrow$ Independent gamma posterior distributions
  $\lambda_i|n_i, t_i \sim \text{Ga}\Big(\alpha \exp\{\mathbf{X}_i'\widehat{\boldsymbol{\beta}}\} + n_i, \alpha + t_i\Big), i = 1, \ldots, k$
- $\Rightarrow$ posterior mean $\dfrac{\alpha \exp\{\mathbf{X}_i'\widehat{\boldsymbol{\beta}}\} + n_i}{\alpha + t_i}, i = 1, \ldots, k$

NONHOMOGENEOUS POISSON PROCESS

- NHPPs characterized by intensity function $\lambda(t)$ varying over time

- $\Rightarrow$ NHPPs useful to describe (*rare*) events whose rate of occurrence evolves over time (e.g. gas escapes in steel pipelines)

  - Life cycle of a new product

    $*$ initial elevated number of failures (*infant mortality*)

    $*$ almost steady rate of failures (*useful life*)

    $*$ increasing number of failures (*obsolescence*)

    $\Rightarrow$ NHPP with a *bathtub* intensity function

- NHPP has no stationary increments unlike the HPP

- Superposition and Coloring Theorems can be applied to NHPPs

- Elicitation of priors raises similar issues as before

INTENSITY FUNCTIONS

Many intensity functions $\lambda(t)$ proposed in literature (see McCollin (ESQR, 2007))

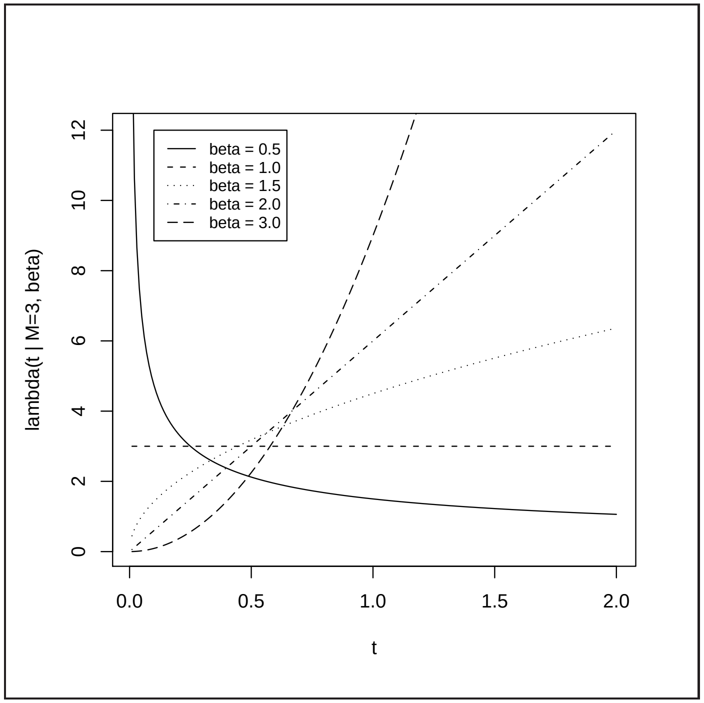
- Different origins
  - Polynomial transformations of HPP constant rate
    * $\lambda(t) = \alpha t + \beta$ (linear ROCOF model)
    * $\lambda(t) = \alpha t^2 + \beta t + \gamma$ (quadratic ROCOF model)
  - Actuarial studies (from hazard rates)
    * $\lambda(t) = \alpha \beta^t$ (Gompertz)
    * $\lambda(t) = \alpha \beta^t + \gamma t + \delta$
    * $\lambda(t) = e^{\alpha + \beta t} + e^{\gamma + \delta t}$
  - Reliability studies
    * $\lambda(t) = \alpha + \beta t + \dfrac{\gamma}{t + \delta}$ (quite close to *bathtub* for adequate values)
    * $\lambda(t) = \alpha \beta (\alpha t)^{\beta - 1} \exp\{\alpha t^\beta\}$ (Weibull software model)

INTENSITY FUNCTIONS

- Different origins
  - Logarithmic transformations
    * $\lambda(t) = \frac{\alpha}{t}$ ($\Rightarrow$ logarithmic $m(t)$)
    * $\lambda(t) = \alpha \log t + \alpha + \beta$
    * $\lambda(t) = \alpha \log(1 + \beta t) + \gamma$
    * $\lambda(t) = \frac{\alpha \log(1 + \beta t)}{1 + \beta t}$ (Pievatolo et al, underground train failures)
  - Associated to distribution functions
    * $\lambda(t) = \alpha f(t;\beta)$, with $f(\cdot)$ density function

NONHOMOGENEOUS POISSON PROCESS

- Different mathematical properties
  - Increasing, decreasing, convex or concave
    - $\lambda(t) = M\beta t^{\beta-1}$, $M, \beta > 0$ (Power Law Process)
    - Different behavior for different βs



NONHOMOGENEOUS POISSON PROCESS

- Different mathematical properties
  - Periodicity (Lewis)
    * $\lambda(t) = \alpha \exp\{\rho \cos(\omega t + \varphi)\}$
    * Earthquake occurrences (Vere-Jones and Ozaki, 1982)
    * Train doors' failures (Pievatolo et al., 2003)
  - Unimodal, starting at 0 and decreasing to 0 when t goes to infinity
    * Ratio-logarithmic intensity
    * $\lambda(t) = \dfrac{\alpha \log(1+\beta t)}{1+\beta t}$
    * Train doors' failures (Pievatolo et al., 2003)

NONHOMOGENEOUS POISSON PROCESS

- Properties of the system under consideration
  - Processes subject to faster and faster (slower and slower) occurrence of events
    $\Rightarrow$ increasing (decreasing) $\lambda(t)$
  - Failures of doors in subway trains, with no initial problems, then subject to an increasing sequence of failures, which later became more rare, possibly because of an intervention by the manufacturer
    $\Rightarrow$ ratio-logarithmic $\lambda(t)$ (Pievatolo et al., 2003)
  - New product $\Rightarrow$ life cycle described by *bathtub* intensity
  - Finite number of bugs to be detected during software testing
    $\Rightarrow$ $m(t)$ finite over an infinite horizon
  - Unlimited number of death in a population
    $\Rightarrow$ $m(t)$ infinite over an infinite horizon (as a good approximation)

CLASSES OF NHPPs

Classes of NHPPs

- Defined through density $f(t)$, with cdf $F(t)$
  - $\lambda(t) = \theta f(t)$ and $m(t) = \theta F(t)$
  - $\Rightarrow \theta$ interpreted as (finite) expected number of events over an infinite horizon
  - Number of bugs in software (Ravishanker et al.)

- Separable intensity $\lambda(t|M, \beta) = Mg(t, \beta)$
  - $M, \beta > 0$
  - g nonnegative function on $[0, \infty)$
  - Very popular intensities:
    * $g(t, \beta) = \beta t^{\beta-1}$ (Power law process)
    * $g(t, \beta) = e^{-\beta t}$ (Cox-Lewis process)
    * $g(t, \beta) = 1/(t + \beta)$ (Musa-Okumoto process)

CLASSES OF NHPPs

- *Musa and Okumoto*: $\lambda(t) \left(= [m(t)]^{'}\right) = \lambda e^{-\theta m(t)}$
  $\Rightarrow m(t) = \dfrac{1}{\theta} \log(\lambda\theta t + 1)$ for $m(0) = 0$
- *PLP*: $\lambda(t) = M\beta t^{\beta-1} \Rightarrow [m(t)]^{'} = \dfrac{\beta m(t)}{t}$
- $\lambda(t) = a(e^{bt} - 1) \Rightarrow [m(t)]^{'} = b[m(t) + at]$
- $\lambda(t) = a \log(1 + bt) \Rightarrow [m(t)]^{'} = \dfrac{b[m(t) + at]}{1 + bt}$
- $\Rightarrow \; [m(t)]^{'} = \dfrac{\alpha m(t) + \beta t}{\gamma + \delta t}$
- $y^{'} = \dfrac{\alpha y + \beta x}{\gamma + \delta x}$
- $\Rightarrow y = e^{\int \alpha/(\gamma+\delta x)dx} \left\{ \int \dfrac{\beta x}{\gamma + \delta x} e^{-\int \alpha/(\gamma+\delta x)dx} dx + c \right\}$

CLASSES OF NHPPs

$m(t)$	$\lambda(t)$
$\frac{t}{\delta}$	$\frac{1}{\delta}$
$\frac{t^2}{2\gamma}$	$\frac{t}{\gamma}$
$\frac{t}{\delta} - \frac{\gamma}{\delta^2} \log\left(1 + \frac{\delta}{\gamma} t\right)$	$\frac{t}{\gamma + \delta t}$
$\lvert c \rvert t^{\alpha/\delta}$	$\lvert c \rvert \frac{\alpha}{\delta} t^{\alpha/\delta - 1}$
$\beta\gamma\left(e^{t/\gamma} - \frac{t}{\gamma} - 1\right)$	$\beta\left(e^{t/\gamma} - 1\right)$
$\frac{\beta}{\delta - 1}\left\{t + \gamma\left[1 - \left(1 + \frac{\delta}{\gamma} t\right)^{1/\delta}\right]\right\}$	$\frac{\beta}{\delta - 1}\left\{1 - \left(1 + \frac{\delta}{\gamma} t\right)^{1/\delta - 1}\right\}$
$\beta\gamma\left(1 + \frac{t}{\gamma}\right) \log\left(1 + \frac{t}{\gamma}\right) - \beta t$	$\beta \log\left(1 + \frac{t}{\gamma}\right)$

CLASSES OF NHPPs

- *Small changes in parameters may imply significative changes in the mathematical expression for* $m(t)$
  - $\alpha = 0, \beta = 1, \gamma = 1, \delta = 2$
    $\Rightarrow m_0(t) = t/2 - (1/4)\log(1 + 2t)$
  - $\alpha > 0, \beta = 1, \gamma = 1, \delta = 2$
    $\Rightarrow m_\alpha(t) = \{t + (1/\alpha)(1 + 2t)^{\alpha/2}\}/(2 - \alpha)$
  - $\Rightarrow \lim_{\alpha \downarrow 0} m_\alpha(t) = m_0(t), \forall t$
- *Open problems*
  - Interpretation
  - Properties of the models (e.g. continuity)
  - Sensitivity and model selection

INFERENCE

- Intensity function of $N(t)$ denoted $\lambda(t|\boldsymbol{\theta})$, $\boldsymbol{\theta}$ parameter

- Events observed at times $T_1 < \ldots < T_n$ in $(0, T]$

- Likelihood $l(\boldsymbol{\theta}|T_1, \ldots, T_n) = \prod_{i=1}^{n} \lambda(T_i|\boldsymbol{\theta}) \cdot e^{-m(T)}$

- Class of NHPPs with $\lambda(t) = \theta f(t|\omega)$

- $\Rightarrow l(\theta, \omega|T_1, \ldots, T_n) = \theta^n \prod_{i=1}^{n} f(T_i|\omega) \cdot e^{-\theta F(T|\omega)}$

  – Exponential distribution Ex(ω): $f(t|\omega) = \omega e^{-\omega t}$ and $F(t|\omega) = 1 - e^{-\omega t}$

  – $\Rightarrow l(\theta, \omega|T_1, \ldots, T_n) = \theta^n \omega^n e^{-\omega \sum_{i=1}^{n} T_i - \theta(1 - \exp\{-\omega T\})}$

INFERENCE

- Likelihood $l(\theta, \omega | T_1, \ldots, T_n) = \theta^n \prod_{i=1}^{n} f(T_i|\omega) \cdot e^{-\theta F(T|\omega)}$
- Independent priors $\theta \sim$ Ga(α, δ) and $f(\omega)$
- $\Rightarrow$ posterior conditionals

$$\begin{aligned} \theta | T_1, \ldots, T_n, \omega &\sim \text{Ga}(\alpha + n, \delta + F(T|\omega)) \\ \omega | T_1, \ldots, T_n, \theta &\propto \prod_{i=1}^{n} f(T_i|\omega) e^{-\theta F(T|\omega)} f(\omega) \end{aligned}$$

- Sample from posterior applying Metropolis step within Gibbs sampler

INFERENCE

- Class of NHPPs with $\lambda(t|M,\beta) = Mg(t,\beta)$
- Likelihood $l(M,\beta|T_1,\ldots,T_n) = M^n \prod_{i=1}^{n} g(T_i,\beta) \cdot e^{-MG(T,\beta)}$
- $G(t,\beta) = \int_0^t g(u,\beta)du$
- Independent priors $M \sim$ Ga(α,δ) and $f(\beta)$
- $\Rightarrow$ posterior conditionals

$$\begin{aligned} M|T_1,\ldots,T_n,\beta &\sim \text{Ga}(\alpha+n,\delta+G(T,\beta)) \\ \beta|T_1,\ldots,T_n,M &\propto \prod_{i=1}^{n} g(T_i,\beta)e^{-MG(T,\beta)}f(\beta) \end{aligned}$$

- Sample from posterior applying Metropolis step within Gibbs sampler

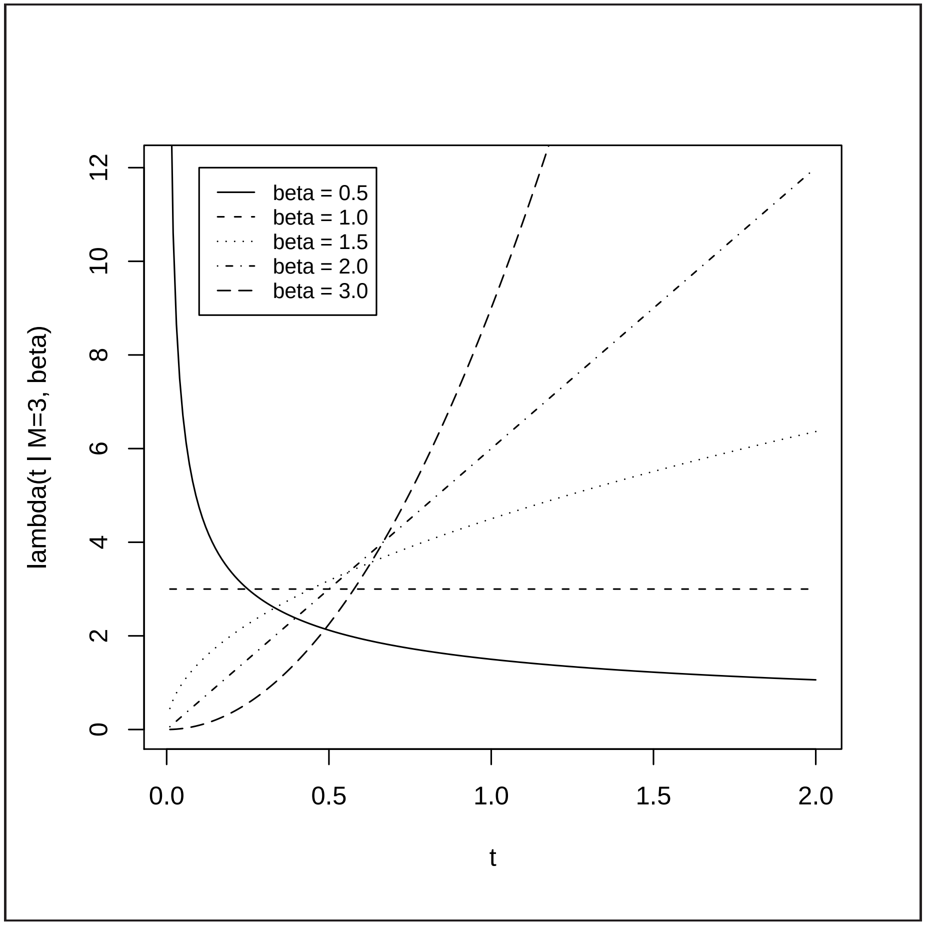
NONHOMOGENEOUS POISSON PROCESS

$N(t)$ Power Law process (PLP) (or Weibull process)

- Two parameterizations:
  - $\lambda(t|\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$ and $m(t;\alpha,\beta) = \left(\frac{t}{\alpha}\right)^{\beta}$, $\alpha,\beta,t > 0$
  - $\lambda(t;M,\beta) = M\beta t^{\beta-1}$ and $m(t;M,\beta) = Mt^{\beta}$, $M,\beta > 0$
  - Link: $\alpha^{-\beta} = M$

- Parameters interpretation
  - $\beta > 1 \Rightarrow$ reliability decay
  - $\beta < 1 \Rightarrow$ reliability growth
  - $\beta = 1 \Rightarrow$ constant reliability
  - $M = m(1)$ expected number of events up to time 1

POWER LAW PROCESS



FREQUENTIST ANALYSIS

Failures $\underline{T} = (T_1, \ldots, T_n) \Rightarrow$ likelihood

$$l(\alpha, \beta \mid \underline{T}) = (\beta/\alpha)^n \prod_{i=1}^{n} (T_i/\alpha)^{\beta-1} \, e^{-(y/\alpha)^\beta}$$

- *Failure truncation* $\Rightarrow y = T_n$

  MLE: $\quad \widehat{\beta} = n / \sum_{i=1}^{n-1} \log(T_n/T_i) \quad$ and $\quad \widehat{\alpha} = T_n / n^{1/\widehat{\beta}}$

  C.I. for β : $\left(\widehat{\beta}\chi^2_{\gamma/2}(2n-2)/(2n), \widehat{\beta}\chi^2_{1-\gamma/2}(2n-2))/(2n)\right)$

- *Time truncation* $\Rightarrow y = T$

  MLE: $\quad \widehat{\beta} = n / \sum_{i=1}^{n} \log(T/T_i) \quad$ and $\quad \widehat{\alpha} = T / n^{1/\widehat{\beta}}$

  C.I. for β : $\left(\widehat{\beta}\chi^2_{\gamma/2}(2n)/(2n), \widehat{\beta}\chi^2_{1-\gamma/2}(2n)/(2n)\right)$

Unbiased estimators, $\widehat{\lambda}(t)$, approx. C.I., hypothesis testing, goodness-of-fit, etc.

BAYESIAN ANALYSIS

Failure truncation $\equiv$ Time truncation

$$l(\alpha, \beta \mid \underline{T}) = (\beta/\alpha)^n \prod_{i=1}^{n} (T_i/\alpha)^{\beta-1} \, e^{-(y/\alpha)^\beta}$$

- $\pi(\alpha, \beta) \propto (\alpha\beta^\gamma)^{-1} \quad \alpha > 0, \beta > 0, \gamma = 0, 1 \Rightarrow \beta|\underline{T} \sim \widehat{\beta}\chi^2_{2(n-\gamma)}/(2n)$
  - Posterior exists, except for $\gamma = 0$ and $n = 1$
  - $\widehat{\beta} = n/\sum_{i=1}^{n} \log(T/T_i)$
  - Posterior mean $\tilde{\beta} = (n - \gamma)/\sum_{i=1}^{n} \log(T/T_i)$
  - Credible intervals easily obtained with standard statistical software
- $\pi(\alpha) \propto \alpha^{-1}$ and $\beta \sim \mathrm{U}(\beta_1, \beta_2) \Rightarrow \pi(\beta|\underline{T}) \propto \beta^{n-1} \prod_{i=1}^{n} T_i^\beta I_{[\beta_1,\beta_2]}(\beta)$
- $\pi(\alpha|\beta) \propto \beta s^{a\beta}\alpha^{-a\beta-1}e^{-b(s/\alpha)^\beta} \quad a, b, s > 0$ and $\beta \sim \mathrm{U}(\beta_1, \beta_2)$

  $$\Rightarrow \pi(\beta|\underline{T}) \propto \beta^n \prod_{i=1}^{n} \left(\frac{T_i}{s}\right)^\beta \left[(\frac{T_n}{s})^\beta + b\right]^{-n-a} I_{[\beta_1,\beta_2]}(\beta)$$

- In all case $\alpha|\underline{T}$ by simulation (but $\alpha|\beta, \underline{T}$ inverse of a Weibull)

BAYESIAN ANALYSIS

Other parametrization

- $l(M, \beta \mid T_1, \ldots, T_n) = M^n \beta^n \prod_{i=1}^{n} T_i^{\beta-1} e^{-MT^\beta}$
- Independent priors $M \sim \mathsf{Ga}(\alpha, \delta)$ and $\beta \sim \mathsf{Ga}(\mu, \nu)$
- Possible dependent prior: $M|\beta \sim \mathsf{Ga}\big(\alpha, \delta^\beta\big)$
- $\Rightarrow$ posterior conditionals (in red changes for dependent prior)

$$\begin{aligned} M|T_1, \ldots, T_n\beta &\sim \mathsf{Ga}\big(\alpha + n, \delta^{\beta} + T^\beta\big) \\ \beta|T_1, \ldots, T_nM &\propto \beta^{\mu+n-1} \exp\{\beta(\sum_{i=1}^{n} \log T_i - \nu) - MT^\beta - M\delta^\beta\} \end{aligned}$$

- Sample from posterior applying Metropolis step within Gibbs sampler

Interest in posterior $\mathcal{E}\beta, \mathcal{P}\{\beta < 1\}$*, modes, C.I.'s,* $\mathcal{E}M$ *(for* $\lambda(t) = M\beta t^{\beta-1}$*)*

BAYESIAN ROBUSTNESS

- Interruption dates for a 115 kV transmission line (Rigdon and Basu, 1989)
- 13 failure dates available, from July 15, 1963 to December 19, 1971
  - 12 failure times, assuming first failure date as time 0
  - Data time truncated on December 31, 1971
- Re-scale failure times so that $[15/7/1963, 31/12/1971]$ becomes $[0, 1]$
- $\Rightarrow$ Factor $e^{-MT^{\beta}}$ in likelihood becomes e^{-M}

  (Simplifying assumption for illustrative purposes)
- Likelihood $l(\beta, M|t_1, \ldots, t_{12}) = \beta^{12}u^{\beta-1}M^{12}e^{-M}$, with

  (re-scaled) failure times t_i's and their product u

BAYESIAN ROBUSTNESS

- Independent priors on M and $\beta \Rightarrow$ independent posterior $\Rightarrow$ focus only on β
- Prior P_0: $\beta \sim \mathrm{Ga}(2, 1.678) \Rightarrow$ prior median $1 \Rightarrow$ reliability growth and decay are equally likely, a priori
- $\Rightarrow 0.722$ posterior mean of β
- ϵ–contamination class of priors around P_0
  - $\Gamma_\epsilon = \{P : P = (1-\epsilon)P_0 + \epsilon Q, Q \in \mathcal{Q}\}$
  - $\epsilon = 0.1$
  - $\mathcal{Q}$ class of all probability measures
- $\sup_{\Gamma_\epsilon} E[\beta|data] = 0.757$ (for Dirac measure at 0.953)
- $\inf_{\Gamma_\epsilon} E[\beta|data] = 0.680$ (for Dirac measure at 0.509)
- Very robust estimates $\Rightarrow$ reliability growth

CHANGE POINTS

- *Bathtub* shaped intensity function $\lambda(t)$ describes life cycle of a new product with an initial decreasing part, a constant part and a final increasing one

- $\lambda(t)$ could be described by the intensity functions of three distinct PLPs
  - First part: $\beta_1 < 1$, M_1
  - Second part: $\beta_2 = 1$, M_2
  - Third part: $\beta_3 > 1$, M_3

- Need to estimate
  - $\beta_1, \beta_2, \beta_3$
  - M_1, M_2, M_3
  - Change points t_1 and t_2

CHANGE POINTS

PLP but valid for previous general class

Changes at each failure time

- Hierarchical Model
  - β_i i.i.d. LN$\big(\phi, \sigma^2\big), i = 0, \ldots, n$
  - $\phi \sim$ N$\big(\mu, \tau^2\big)$
  - $\sigma^2 \sim$ IGa(ρ, γ)
- Gamma prior for M
- Conditional posteriors
  - Gamma for M
  - Inverse Gamma for σ^2
  - Normal for ϕ
  - Known (apart from a constant) for β_i's

$\Rightarrow$ Metropolis-Hastings and Gibbs sampling

CHANGE POINTS

Changes at each failure time

- Dynamic Model
  - $\log \beta_i = \log a + \log \beta_{i-1} + \epsilon_i, i = 1, \ldots, n$
  - $\epsilon_i \sim \mathrm{N}\big(0, \sigma^2\big)$
- Priors
  - Gamma for M
  - Inverse Gamma for σ^2
  - Lognormal for $a \mid \sigma^2$
  - Lognormal for $\beta_0 \mid \sigma^2$
- Conditional posteriors
  - Gamma for M
  - Inverse Gamma for σ^2
  - Lognormal for a
  - Known (apart from a constant) for β_i's

$\Rightarrow$ Metropolis-Hastings and Gibbs sampling

CHANGE POINTS

Changes at a random number of failures

- Dynamic model as before
- Bernoulli r.v.'s for change/no change
- Beta priors on Bernoulli parameter

Changes at a random number of points
$\Rightarrow$ Reversible jump MCMC with steps:

- change of M and β at a randomly chosen change point
- change to the location of a randomly chosen change point
- "birth" of a new change point at a randomly chosen location in $(0, y]$
- "death" of a randomly chosen change point

COAL-MINING DISASTERS

Ruggeri and Sivaganesan (2005)

- Dates of British serious coal-mining disasters, between 1851 and 1962: a well-known data set for change point analysis
- RJMCMC to find change points
- Posterior probabilities for number of change points

k	0	1	2	3
Prob.	0.01	0.85	0.14	0.09

- Strong evidence in favor of one point (like in Raftery and Akman, 1986) but some weak evidence for 2 points
- March 1892: posterior median of the change point (conditional on having only a single change point)
- April 1886 - June 1896: 95% equal tail credible interval

COMPOUND POISSON PROCESSES

- In Poisson processes events occur individually $\Rightarrow$ sometimes a limitation
- Batch arrivals
  - passengers exiting a bus at a bus stop
  - arrival of multiple claims to an insurance company
- $\Rightarrow$ Compound Poisson process (see, e.g., Snyder and Miller, 1991), generalizes the Poisson process to allow for multiple arrivals
  - $N(t)$ Poisson process with intensity $\lambda(t)$ (*center of clusters*)
  - Sequence of i.i.d. random variables $\{Y_i\}$, independent of $N(t)$ (*size of jump*)
  - $\Rightarrow$ Compound Poisson process: counting process (*jump process*) defined by
    * $S(t) = \sum_{i=1}^{N(t)} Y_i$,
    * with $S(t) = 0$ when $N(t) = 0$

COMPOUND POISSON PROCESSES

- $S(t)$ compound Poisson process $\Rightarrow$ for $t < \infty$

  1. $E[S(t)] = E\left[E\left[\sum_{i=1}^{n} Y_i | N(t) = n\right]\right] = m(t)E[Y_i]$,

  2. $V[S(t)] = E\left[V\left[\sum_{i=1}^{n} Y_i | N(t) = n\right]\right] + V\left[E\left[\sum_{i=1}^{n} Y_i | N(t) = n\right]\right]$
  $= E[N(t)]V[Y_i] + V[N(t)](E[Y_i])^2 = m(t)E\left[Y_i^2\right]$.

- Inference for compound Poisson processes $\Rightarrow$ complex task, as shown by a simple example

  – $N(t)$: HPP with rate λ

  – $\{Y_i\}$: i.i.d. exponential Ex(μ) r.v.'s

COMPOUND POISSON PROCESSES

- Suppose $S(t) = s$ observed, with $t, s > 0$
- Likelihood from

$$\begin{aligned} f(S(t)=s) &= \sum_{n=1}^{\infty} f\left(\sum_{i=1}^{n} Y_i = s | N(t) = n\right) P(N(t) = n) \\ &= \sum_{n=1}^{\infty} \frac{\mu^n}{n-1!} s^{n-1} e^{-\mu s} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \end{aligned}$$

- since sum of n i.i.d. exponential Ex$(\mu) \Rightarrow$ Ga(n, μ)
- Gamma priors Ga(α, β) and Ga(γ, δ) for μ and λ, respectively
- $\Rightarrow$ posterior

$$f(\mu, \lambda | S(t) = s) \propto \sum_{n=1}^{\infty} \frac{s^{n-1}}{n-1!} \frac{\beta^\alpha}{\Gamma(\alpha)} \mu^{\alpha+n-1} e^{-(\beta+s)\mu} \cdot \frac{t^n}{n!} \frac{\delta^\gamma}{\Gamma(\gamma)} \lambda^{\gamma+n-1} e^{-(\delta+t)\lambda},$$

COMPOUND POISSON PROCESSES

- Integrating w.r.t. $\mu \Rightarrow$ posterior on λ

$$f(\lambda|S(t)=s) \propto \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\beta^{\alpha}}{(\beta+s)^{\alpha+n}} \frac{s^{n-1}}{n-1!} \cdot \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \frac{\delta^{\gamma}}{(\delta+t)^{\gamma+n}} \frac{t^{n}}{n!} g(\gamma+n, \delta+t),$$

  with $g(a,b)$ density function of a gamma Ga(a,b) random variable

- Similar for μ, integrating w.r.t. λ

- Multiple observations $\Rightarrow$ very cumbersome computations

- $E[S(T)] = \int m(t|\lambda) E[Y_i|\mu] f(\mu,\lambda|S(t)=s) d\mu d\lambda$

COMPOUND POISSON PROCESSES

- Widely used to model insurance claims
- $X(t) = X(0) + ct - \sum_{i=1}^{N(t)} Y_i$
  - $X(0)$ initial insurer's reserve
  - $X(t)$ insurer's reserve at time t
  - c constant premium (paid to insurer) rate
  - $N(t)$ number of claims up to time t
  - Y_i, $i = 1, \dots, N(t)$, amount of i-th claim
- Interest in ruin probability

MODULATED POISSON PROCESS

- Simple extension of Poisson process: introduction of covariates in $\lambda(t)$
  - Masini et al (2006): rate λ of HPP multiplied by factors $\mu_i^{X_{ji}}$ depending on covariates X_{ji} taking values 0 or 1

- Events occur according to a modulated Poisson process if
  - $\lambda_i(t) = \lambda_0(t)e^{\mathbf{X}_i'\boldsymbol{\beta}}$ for $i = 1, \ldots, n$, with
  - $\lambda_0(t)$ baseline intensity function
  - $\mathbf{X}_i = (X_{i1}, \ldots, X_{im})$, $i = 1, \ldots, n$ different combinations of covariates
  - m-variate parameter $\boldsymbol{\beta}$

INFERENCE

- Each combination of covariates produces a Poisson process $N_i(t)$
- Superposition Theorem $\Rightarrow$ unique Poisson process $N(t)$
- Bayesian inference for general modulated Poisson processes very similar to one for HPP, with the addendum of a distribution over the parameter $\boldsymbol{\beta}$
- Call center arrival data (Soyer and Tarimcilar, 2008, and Landon et al, 2010)
  - Calls typically linked to individual advertisements
  - Interest in evaluating the efficiency of advertisements
  - X_i vector of covariates describing characteristics of i-th advertisement
    - media expenditure (in $'s)
    - venue type (monthly magazine, daily newspaper, etc.)
    - ad format (full page, half page, color, etc.)
    - offer type (free shipment, payment schedule, etc.)
    - seasonal indicators

SELF-EXCITING PROCESSES

- In a Poisson process occurrence of events does not affect intensity function at later times

- Such property not always realistic
  - Sequence of aftershocks after major earthquake
  - Introduction of new bugs during software testing and debugging

- $\Rightarrow$ Self-exciting process (SEP) introduced to describe phenomena in which occurrences affect next ones (Snyder and Miller, 1991, Ch. 6, and Hawkes and Oakes, 1974)

- Deterministic intensity function of NHPP

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t] \geq 1)}{\Delta t}$$

SELF-EXCITING PROCESSES

- Intensity process associated to SEP, dependent on history

$$\lambda(t) = \mu(t) + \sum_{j=1}^{N(t^-)} g_j(t - T_j),$$

  - $\mu(t)$ deterministic function
  - T_j's occurrence times
  - g_j's nonnegative functions expressing influence of past observations on the intensity process

- Likelihood function formally similar to NHPP's

  - $P(T_1, \ldots, T_n) = \prod_{i=1}^{n} \lambda(T_i) \cdot e^{-\int_0^T \lambda(t)dt}$, with
  - $T_1 < \ldots < T_n$ arrival times in $(0, T]$

SELF-EXCITING PROCESSES

- New tyre of a bicycle goes flat according to an HPP with rate λ
  - No degradation over $[0, T]$, if not punctured
  - Punctures occur *randomly*
- Flat tyre occurs at $T_1 < \ldots < T_n$ over $[0, T]$
  - Tyre repaired at each T_i but more prone to new failures
  - $\Rightarrow$ Add μ_i to previous rate of HPP
  - $\Rightarrow$ Stepwise HPP, very simple example of SEP, with $\lambda(t) = \lambda + \sum_{i=1}^{N(t^-)} \mu_i I_{(T_i,\infty)}(t)$
- Likelihood $\prod_{i=1}^{n} \left(\lambda + \sum_{j=1}^{i-1} \mu_j \right) e^{-\lambda T - \sum_{i=1}^{n} \mu_i (T - T_i)}$
- Gamma priors on λ and μ_i's
- $\Rightarrow$ Mixtures of gamma distributions as full conditional posteriors

DOUBLY STOCHASTIC POISSON PROCESSES

- Intensity function of a self-exciting process $\Rightarrow$ (random) intensity process
  - dependent on $N(t)$ itself
  - paths known when observing events in $N(t)$

- Doubly stochastic Poisson process or Cox process (Cox, 1955)
  - Extension of Poisson processes, allowing for unknown paths of the intensity process, when only $N(t)$ is given
  - Two step randomization procedure:
    process $\Lambda(t)$ used to generate another process $N^*(t)$ acting as its intensity
    * $N(t)$ is a Poisson process on $[0, \infty)$
    * $\Lambda(t)$ stochastic process, independent from $N(t)$, with nondecreasing paths, s.t. $\Lambda(0) \geq 0$
    * $\Rightarrow N^*(t) = N(\Lambda(t))$ doubly stochastic Poisson process

DOUBLY STOCHASTIC POISSON PROCESSES

- Definition $\Rightarrow N(t)$ Poisson process conditional on sample path $\lambda(t)$ of process $\Lambda(t)$
- $\lambda(t)$ deterministic $\Rightarrow N(t)$ Poisson process
- $\Lambda(t) \equiv \Lambda$ r.v. $\Rightarrow$ mixed Poisson process
- Very few papers on Bayesian analysis of doubly stochastic Poisson processes
  - Gutiérrez-Peña and Nieto-Barajas (2003) modelled $\Lambda(t)$ with a gamma process
  - Varini and Ogata (forthcoming) on seismic applications
- Problem: repeated observations (paths) of the process needed to estimate intensity process and avoid indistinguishability from a NHPP based on a single path

MARKED POISSON PROCESSES

- Points of Poisson process might be labeled with some extra information
- Observations become pairs (T_i, m_i)
  - T_i occurrence time
  - m_i (*mark*) outcome of an associated random variable
- Thinning (or Coloring Theorem)
  - Equivalent to introducing a mark m valued $\{1, \ldots, n\}$ and
  - assigning the event to the class A_m in the family of mutually exclusive and exhaustive classes $\{A_1, \ldots, A_n\}$

EARTHQUAKE OCCURRENCES

- Earthquakes as point events subject to randomness

- Earthquakes occurrences often modeled as realizations of a point process, since Vere-Jones (1970) (see Vere-Jones, 2011, for an account of the history of stochastic models used in analyzing seismic activities)

- Some stochastic processes
  - NHPP with $\lambda(t) = \alpha \exp\{\rho \cos(\omega t + \varphi)\}$ (Vere-Jones and Ozaki, 1982)
  - Marked Poisson processes used to jointly model occurrence and magnitude of earthquakes (Rotondi and Varini, 2003)
  - Stress release model to analyze data in the Italian Sannio-Matese-Ofanto-Irpinia region (considered later) (Rotondi and Varini, 2007, but model introduced by Vere-Jones, 1978)
    - Justified by Reid's physical theory: stress in a region accumulates slowly over time, until it exceeds the strength of the medium, and, then, it is suddenly released and an earthquake occurs

EARTHQUAKE OCCURRENCES

- Data from Sannio Matese, area in southern Italy subject to a consistent, sometimes very disruptive, seismic activity
  - 6.89 magnitude earthquake on 23/11/1980 in Irpinia caused many casualties and considerable damage
- Shocks in Sannio Matese since 1120 catalogued exhaustively in Postpischl (1985), using current and historical data such as church records
- For each earthquake, the catalogue contains many data
  - Occurrence time (often up to the precise second)
  - Latitude and longitude
  - Intensity (strength of shaking at location as determined from effects on people and environment)
  - Magnitude (energy released at the source of the earthquake, measured by seismographs or, earlier, computed from intensity)
  - Name of the place of occurrence

EARTHQUAKE OCCURRENCES

- Exploratory data analysis identified three different behaviors of the occurrence time process since 1120 (Ladelli et al, 1992)
- More formal analysis $\Rightarrow$ change-point model as before
- Consider earthquakes in the third period, i.e. from 1860 up to 1980
- Presence of foreshocks and aftershocks, sometimes hardly recognized $\Rightarrow$ consider one earthquake, the strongest, as the main shock in any sequence lasting one week
- Sannio Matese divided into three sub-regions, relatively homogeneous from a geophysical viewpoint

EARTHQUAKE OCCURRENCES

- Marked Poisson model (occurrence time and magnitude)
- X interoccurrence times (in years) of a major earthquake (i.e. with magnitude not smaller than 5)
  - First interoccurrence time given by elapsed time between first and second earthquake
- $Y^{'}$ magnitude of a major earthquake
- Z number of minor earthquakes occurred in a given area since previous major one
- Earthquakes occur according to an HPP
- Each earthquake has probability p of being a major earthquake (and $1 - p$ of being a minor one)

EARTHQUAKE OCCURRENCES

- Coloring Theorem $\Rightarrow$ decompose the Poisson process into two independent processes with respective rates λp (major) and $\lambda(1-p)$ (minor)

- $\Rightarrow$ Interoccurrence times $X \sim$ Ex(λp)

- $\Rightarrow$ Conditionally on time x (realization of X), $Z \sim$ Po$(\lambda(1-p)x)$

- $\Rightarrow Z \sim$ Ge(p) since, for $z \in \mathbb{N}$,

$$
\begin{aligned}
P(Z = z) &= \int_0^\infty P(Z = z | X = x) f(x) dx \\
&= \int_0^\infty e^{-\lambda(1-p)x} \frac{[\lambda(1-p)x]^z}{z!} \cdot \lambda p e^{-\lambda p x} dx = p(1-p)^z
\end{aligned}
$$

EARTHQUAKE OCCURRENCES

- Magnitude $Y^{'}$ independent on X and Z
- Although continuous, model $Y^{'}$ as a discrete r.v. which gets the values $(5, 5.1, 5.2, \ldots)$ (one decimal, in general, in data recorded in the earthquakes catalogue)
- Actually consider $Y = 10(Y^{'} - 5) \sim \text{Ge}(\mu)$
- Approximation works well for $\mu \simeq 1$
  - $\Rightarrow \sum_{k=K}^{\infty} \mu(1-\mu)^k = (1-\mu)^K \simeq 0$, even for small K
  - $\Rightarrow$ Quite small probability of Y being large
- Joint density of (X, Y, Z) given by

$$f(x,y,z) = f(x)\, P(Y=y)\, P(Z=z|X=x) = \lambda p e^{-\lambda x} \frac{[\lambda(1-p)x]^z}{z!} \mu(1-\mu)^y$$

- $\Rightarrow$ Likelihood, for n observations $\{X_i, Y_i, Z_i\}$

$$l(p,\lambda,\mu|data) = \lambda^{n+\sum Z_i} e^{-\lambda \sum X_i} p^n (1-p)^{\sum Z_i} \mu^n (1-\mu)^{\sum Y_i} \prod_{i=1}^{n} \frac{X_i^{Z_i}}{Z_i!}$$

EARTHQUAKE OCCURRENCES

- Independent priors
  - $p \sim \text{Be}(\alpha_1, \beta_1)$
  - $\lambda \sim \text{Ga}(\alpha_2, \beta_2)$
  - $\mu \sim \text{Be}(\alpha_3, \beta_3)$

- $\Rightarrow$ Independent posterior distributions
  - $p|\text{data} \sim \text{Be}(n + \alpha_1, \sum Z_i + \beta_1)$
  - $\lambda|\text{data} \sim \text{Ga}(n + \sum Z_i + \alpha_2, \sum X_i + \beta_2)$
  - $\mu|\text{data} \sim \text{Be}(n + \alpha_3, \sum Y_i + \beta_3)$

EARTHQUAKE OCCURRENCES

- Posterior mean
  - $E[p|\text{data}] = \dfrac{n+\alpha_1}{n+\alpha_1+\sum Z_i+\beta_1}$
  - $E[\lambda|\text{data}] = \dfrac{n+\sum Z_i+\alpha_2}{\sum X_i+\beta_2}$
  - $E[\mu|\text{data}] = \dfrac{n+\alpha_3}{n+\alpha_3+\sum Y_i+\beta_3}$

EARTHQUAKE OCCURRENCES

- Predictive densities for X_{n+1}, Y_{n+1} and Z_{n+1}

$$f(x_{n+1}|\text{data}) = \frac{(n+\sum Z_i + \alpha_2)(\sum X_i + \beta_2)^{n+\sum Z_i+\alpha_2}\Gamma(n+\alpha_1+\sum Z_i+\beta_1)}{\Gamma(n+\alpha_1)\Gamma(\sum Z_i+\beta_1)} \times$$

$$\times \int_0^1 \frac{p^{n+\alpha_1}(1-p)^{\sum Z_i+\beta_1-1}}{(px_{n+1}+\sum X_i+\beta_2)^{n+\sum Z_i+\alpha_2+1}} dp,$$

$$P(Y_{n+1}=y_{n+1}|\text{data}) = \frac{(n+\alpha_3)\Gamma(n+\alpha_3+\sum Y_i+\beta_3)\Gamma(\sum Y_i+\beta_3+y_{n+1})}{\Gamma(\sum Y_i+\beta_3)\Gamma(n+\alpha_3+\sum Y_i+\beta_3+y_{n+1}+1)},$$

$$P(Z_{n+1}=z_{n+1}|\text{data}) = \frac{(n+\alpha_1)\Gamma(n+\alpha_1+\sum Z_i+\beta_1)\Gamma(\sum Z_i+\beta_1+z_{n+1})}{\Gamma(\sum Z_i+\beta_1)\Gamma(n+\alpha_1+\sum Z_i+\beta_1+z_{n+1}+1)},$$

for $x_{n+1} \geq 0$ and $y_{n+1}, z_{n+1} \in \mathbb{N}$

EARTHQUAKE OCCURRENCES

- $p \sim$ Be(α_1, β_1), $\lambda \sim$ Ga(α_2, β_2) and $\mu \sim$ Be(α_3, β_3) independent a priori
- p: probability that, given an earthquake occurs, it is a major one
  - Major earthquakes very unlikely w.r.t. minor ones $\Rightarrow$ assume $E[p]$ close to 0
  - $\alpha_1 = 2$ and $\beta_1 = 8 \Rightarrow E[p] = 1/5$
- Major earthquake occurs, on average, every ten years
  - $\Rightarrow E[X] = 10 = 1/(\lambda p)$
  - $\alpha_2 = 2$ and $\beta_2 = 4 \Rightarrow E[\lambda] = 1/2 \Rightarrow E[\lambda]E[p] = 1/10$
- As discussed earlier, μ very close to 1
  - $\alpha_3 = 8$ and $\beta_3 = 2 \Rightarrow E[\mu] = 4/5$
- Prior variances denote strong beliefs:
  $V[p] = 4/275$, $V[\lambda] = 1/8$ and $V[\mu] = 4/275$

EARTHQUAKE OCCURRENCES

Number and sums of observations in three areas in Sannio Matese

	n	$\sum x_i$	$\sum y_i$	$\sum z_i$
zone 1	3	50.1306	9	713
zone 2	16	81.6832	53	1034
zone 3	14	118.9500	100	812

Parameters of posterior distributions

	$p \sim$ Be	$\lambda \sim$ Ga	$\mu \sim$ Be
zone 1	(5,721)	(718,54.1306)	(11,11)
zone 2	(18,1042)	(1052,85.6832)	(24,55)
zone 3	(16,820)	(828,122.9500)	(22,102)

EARTHQUAKE OCCURRENCES

Posterior expectations (and standard deviations)

	$E[p\|\text{data}]$	$E[\lambda\|\text{data}]$	$E[\mu\|\text{data}]$
zone 1	.0069	13.2642	.5000
	(.0031)	(.4950)	(.1043)
zone 2	.0170	12.2778	.3038
	(.0040)	(.3785)	(.0514)
zone 3	.0191	6.7344	.1774
	(.0047)	(.2340)	(.0342)

95% Credible intervals

	$p\|\text{data}$	$\lambda\|\text{data}$	$\mu\|\text{data}$
zone 1	(.0022,.0141)	(12.3116,14.2518)	(.2978,.7022)
zone 2	(.0101,.0256)	(11.5470,13.0307)	(.2081,.4089)
zone 3	(.0110,.0295)	(6.2835,7.2008)	(.0976,.2128)

EARTHQUAKE OCCURRENCES

- Percentage of major earthquakes very small and not significantly different for the three zones, although relatively few major earthquakes occur in zone 1 (smallest p and $p\lambda$, besides a small number of shocks)
  - Posterior density of each p very concentrated around its mean ($\simeq .01$)
  - For all p's $\Rightarrow$ probability larger than 95% of being in the interval $(.0022, .0295)$
- Quite different occurrence rate in the three zones, as λ has very different means and credible intervals: highest rates in zones 1 and 2
- Few minor earthquakes in zone 3, but major earthquakes characterized by larger magnitudes (small μ) $\Rightarrow$ most disruptive earthquakes occur mainly in zone 3
- Zone 2 and 3 with similar number of major earthquakes, but longer interoccurrence times, smaller number of minor earthquakes and larger magnitudes in zone 3
  - Does greater elapsed times between shocks imply greater magnitudes?

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

- Motivated by problems of vibration in commercial aircrafts that cause fatigue in materials, Birnbaum and Saunders (1969) introduced a probability distribution (BS) that describes lifetimes of specimens exposed to fatigue due to cyclic stress

- BS model is based on a physical argument of cumulative damage that produces fatigue in materials, considering the number of cycles under stress needed to force a crack extension due to fatigue to grow beyond a threshold, provoking the failure of the material

- Density and cdf of BS distribution given by

$$f(x) = \frac{\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}}}{2\alpha x} \phi\left(\frac{\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}}{\alpha}\right) \text{ and } F(x) = \Phi\left(\frac{\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}}{\alpha}\right).$$

  with $x > 0$, $\alpha, \beta > 0$, and ϕ and Φ standard Gaussian pdf and cdf

- Several extensions and generalizations of the BS distribution

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

- NHPP $N(t)$ number of cycles occurring in $[0, t]$
  - Intensity function $\lambda(t)$, with $\lambda(t) > 0, \forall t$
  - Mean value function $\Lambda(t)$ strictly increasing
- $\{\xi_k;\ k \in \mathbb{N}\}$ sequence of i.i.d. r.v.'s denoting crack extension due to kth stress cycle
  - Mean μ and variance σ^2
- $\Rightarrow$ compound Poisson $W_t = \sum_{k=1}^{N_t} \xi_k$: total crack extension produced during $[0, t]$
- Interest in $W_t > \omega^*$ since failure occurs for crack length exceeding a critical value ω^*
- $T = \texttt{inf}\{t > 0\colon W_t > \omega^*\}$ lifetime until failure occurs
- $\Rightarrow$ Interest in deriving the distribution of T

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

Theorem

- $\{R_t; t \geq 0\}$ process defined as

$$R_t = \frac{\sum_{k=0}^{N_t} \xi_k - \mu\Lambda(t)}{[\{\mu^2 + \sigma^2\}\Lambda(t)]^{1/2}}$$

- $\Lambda(t) \underset{t\to\infty}{\longrightarrow} \infty$

- $\Rightarrow R_t \xrightarrow[t\to\infty]{\mathcal{D}} R$
  - $R \sim \mathrm{N}(0, 1)$
  - $\mathcal{D}$ convergence in distribution

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

- T and W_t related:

$$\{T \leq t\} = \{W_t \geq \omega^*\} = \left\{ \frac{\sum_{k=1}^{N_t} \xi_k - \mu\Lambda(t)}{\left[\{\mu^2 + \sigma^2\}\Lambda(t)\right]^{1/2}} \geq \frac{\omega^* - \mu\Lambda(t)}{\left[\{\mu^2 + \sigma^2\}\Lambda(t)\right]^{1/2}} \right\}$$

- Combined with Central Limit Theorem $\Rightarrow$ approximate, for t *large enough*

$$\mathbb{P}(W_t \geq \omega^*) = \mathbb{P}(T \leq t) \approx \Phi\left(\frac{\mu\sqrt{\Lambda(t)}}{[\mu^2 + \sigma^2]^{1/2}} - \frac{\omega^*}{[\mu^2 + \sigma^2]^{1/2}\sqrt{\Lambda(t)}} \right)$$

- As in Birnbaum and Saunders, approximation treated as exact

- $\Rightarrow$ Λ-BS distribution Λ-$\mathrm{BS}(\alpha, \beta, \Lambda)$, with cdf

$$F_T(t) = \mathbb{P}(T \leq t) = \Phi\big(\big[\sqrt{\Lambda(t)/\beta_\Lambda} - \sqrt{\beta_\Lambda/\Lambda(t)}\big]/\alpha\big),\ t > 0,$$

with $\alpha = \sqrt{[\mu^2 + \sigma^2]/[\omega^*\mu]}$ and $\beta_\Lambda = \omega^*/\mu$

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

- Link with a Gaussian r.v. Z

$$Z = \frac{1}{\alpha}\left[\sqrt{\Lambda(T)/\beta_\Lambda} - \sqrt{\beta_\Lambda/\Lambda(T)}\right] \sim \mathrm{N}(0,1)$$

$$\Leftrightarrow T = \Lambda^{-1}\Big(\beta_\Lambda\big[\alpha Z/2 + \sqrt{\{\alpha Z/2\}^2 + 1}\big]^2\Big) \sim \Lambda - \mathrm{BS}(\alpha, \beta, \Lambda)$$

- Pdf of $T \sim \Lambda$-$\mathrm{BS}(\alpha,\beta,\Lambda)$ given by $f_T(t) = \phi\left(a_t\right) A_t$, for $t > 0$, $\alpha > 0$ and $\beta > 0$, where

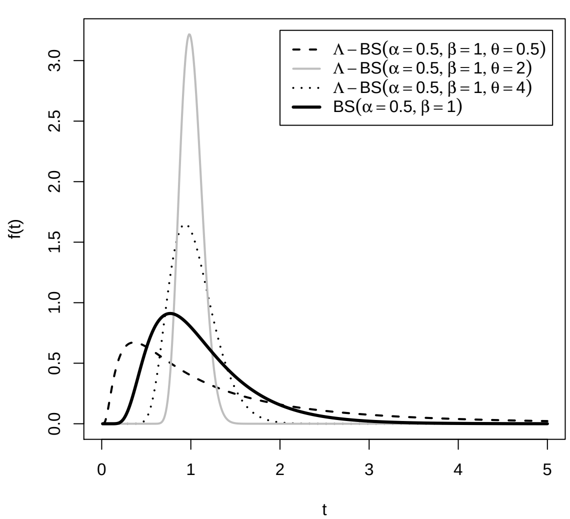
$$a_t = a_t(\alpha,\beta,\Lambda) = \frac{1}{\alpha}\left[\sqrt{\frac{\Lambda(t)}{\beta_\Lambda}} - \sqrt{\frac{\beta_\Lambda}{\Lambda(t)}}\right], \quad A_t = \frac{\mathrm{d}}{\mathrm{d}t}a_t = \frac{\Lambda'(t)\left[\Lambda(t) + \beta_\Lambda\right]}{2\alpha\sqrt{\beta_\Lambda}\Lambda(t)^{3/2}}$$

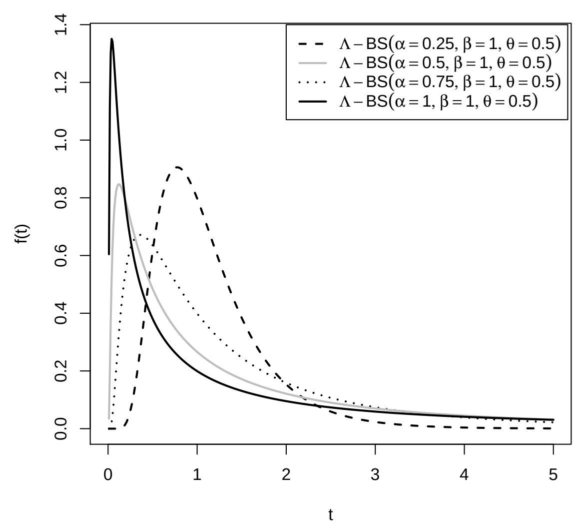
- BS distribution obtained for HPP with $\lambda = 1$

A NEW BIRNBAUM-SAUNDERS DISTRIBUTION

- PLP: $\lambda(t) = M\theta t^{\theta-1}$
- $\Rightarrow$ pdf $f_T(t) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\alpha^2}\left[\left\{\dfrac{t}{\beta}\right\}^{\theta} + \left\{\dfrac{\beta}{t}\right\}^{\theta} - 2\right]\right) \dfrac{\theta}{2\alpha\, t}\left[\left\{\dfrac{t}{\beta}\right\}^{\theta/2} + \left\{\dfrac{\beta}{t}\right\}^{\theta/2}\right]$
- Pdf does not depend on M

- Plots of Λ-BS densities for some α, β and θ





OTHER ISSUES

- Spatio-temporal models, especially spatial point processes, including Poisson ones, are getting more and more popular, mostly stemming from environmental and epidemiological problems (Banerjee et al, 2004)
- (Extended) gamma process conjugate prior on the intensity function for data coming from replicates of a Poisson process (Lo, 1982)
- Intensity function of a spatial NHPP modeled with a Bayesian nonparametric mixture (Kottas and Sansò, 2007)
- Under the Bayesian nonparametric approach, intensity function seen as a realization from a process $\Rightarrow$ data viewed as arising from a doubly stochastic Poisson process
- de Miranda and Morettin (2011) used wavelet expansions to model the intensity function in a classical framework $\Rightarrow$ possible Bayesian approach